



Lois de comportement pour le calcul de structures sous chargement cyclique

Choix et influence sur les réponses asymptotiques

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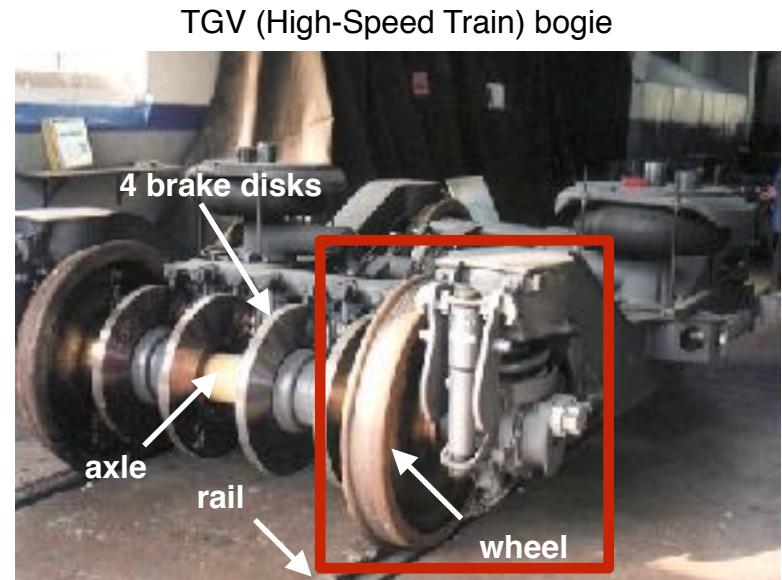
Choices and convictions

- A plasticity course:
 - The topics? Solid state physics? Metallurgy? **Mechanics**
 - The modeling scale? Dislocations? Crystals? Polycrystals?
« **Homogeneous** » **materials and structures**
 - The objective? A fine description of physical phenomena? **Structural computations under cyclic loadings**
- Some strong assumptions:
 - (Generally), asymptotic responses are necessary and **sufficient** for a fatigue analysis of structures
 - If possible, asymptotic response has to be **directly estimated**
 - This requires a **complete** and **consistent strategy: options must be chosen**
 - Other options are possible ;-)

Guiding example

■ Train component: **bogie**

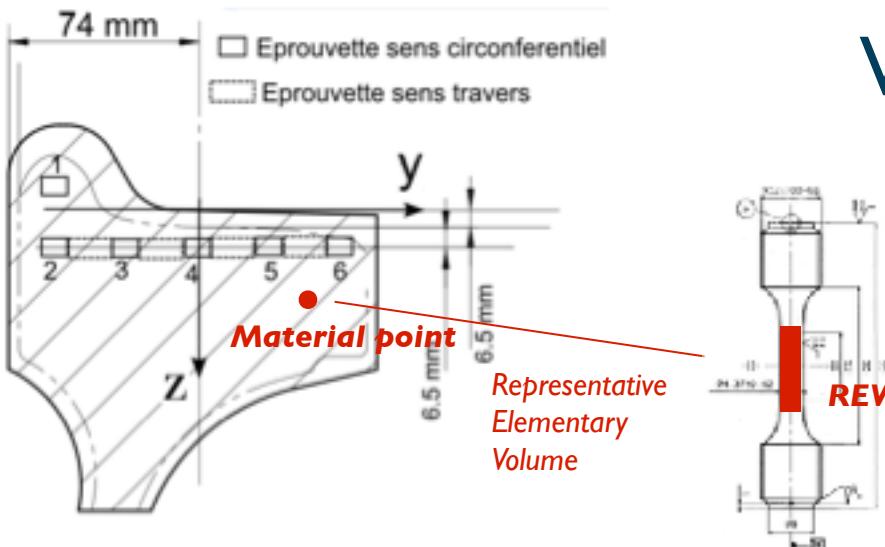
- complex loadings
- “classical” structures:
 - wheel, rail, wheel axle
 - 150 years of industrial progress
 - continuous improvements
- steels: well known material



■ Illustration:

- **residual stresses** due to the manufacturing process (quenching)
- **asymptotic regimes** under cyclic loadings (in particular shakedown)
- **fatigue** design: see next part with Thierry

Wheel: material scale



Non linear kinematic hardening

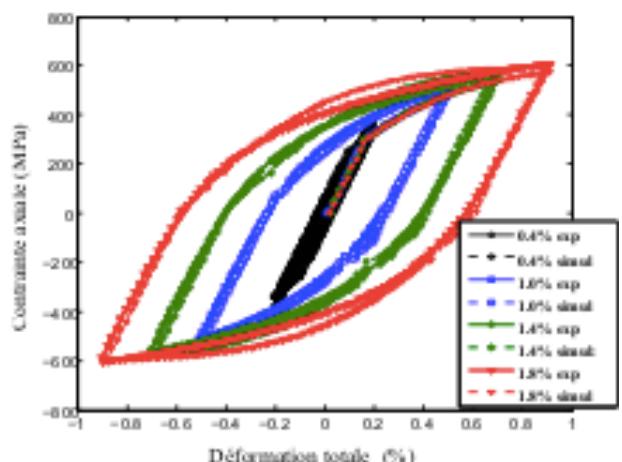
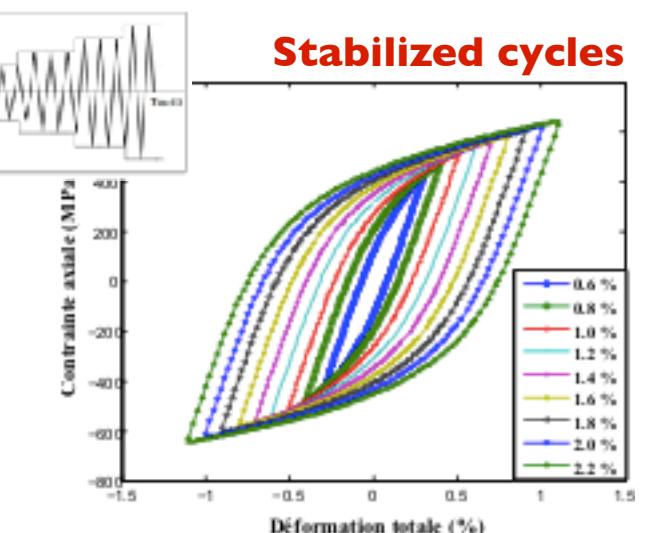
$$f(\underline{\sigma}) = \sqrt{\frac{3}{2}(\underline{s} - \underline{\underline{X}}) : (\underline{s} - \underline{\underline{X}})} - \sigma_0$$

$$\dot{\underline{\underline{X}}} = \frac{2}{3} H \dot{\underline{\varepsilon}}_p - \frac{2}{3} \gamma \underline{\underline{X}} \dot{p}$$

$$\dot{p} = \sqrt{\frac{2}{3} \dot{\underline{\varepsilon}}_p : \dot{\underline{\varepsilon}}_p}$$

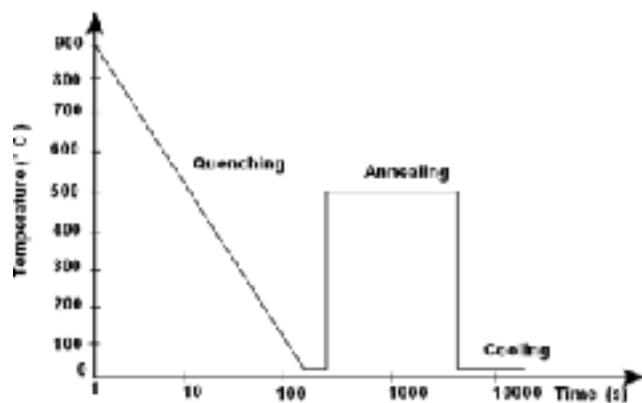
available in
ABAQUS

Parameters: calibrated on the stabilized cycles





Wheels: forging and quenching

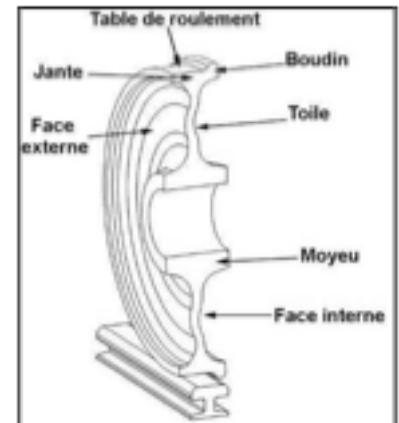
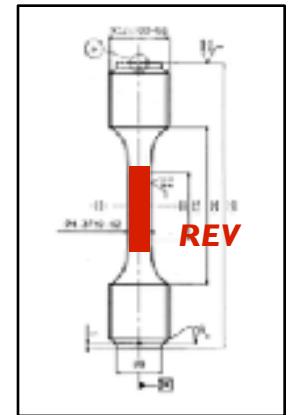


Wheel: structure scale

■ Structural computations:

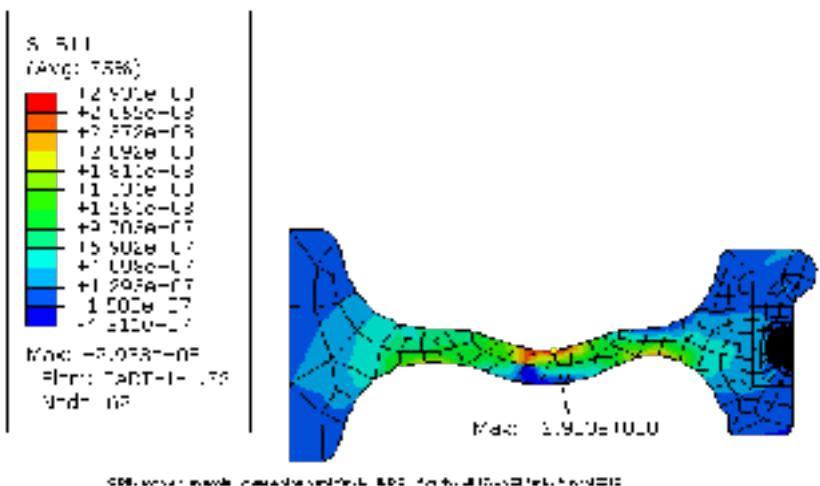
- From the previous step: **REV's constitutive law = material**
- From geometry+BC (+ loading) : **structure**
 - step 1: **residual stress** estimation (quenching: temperature loading)
 - step 2 : **cyclic loading** due to contact (local stress paths, shakedown or not)

Material



Structure

Radial stresses



Residual stress simulations

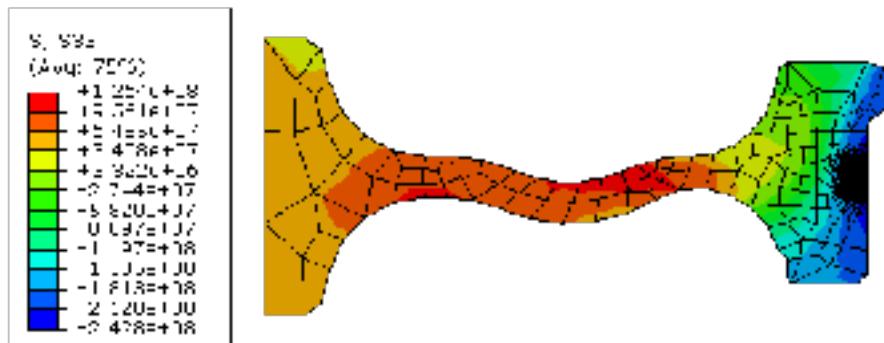
Plasticity with
non linear kinematic hardening:

- between RT and 1000°C
- monotonic curves

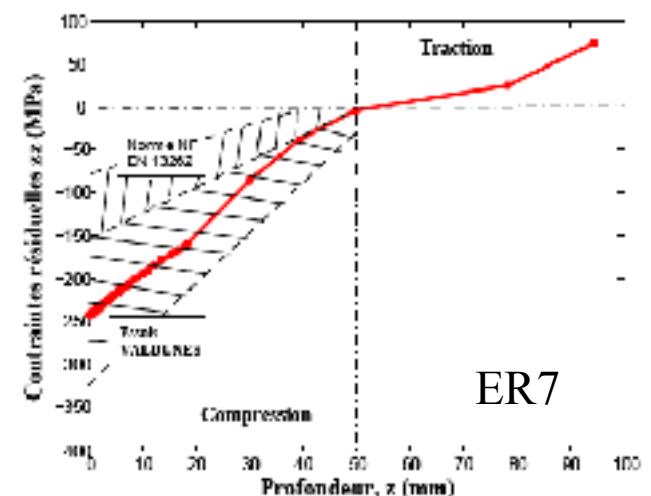
[Brunel, 2007]

Maximum value in the flange:
293 Ma (in tension)

Circumferential stresses

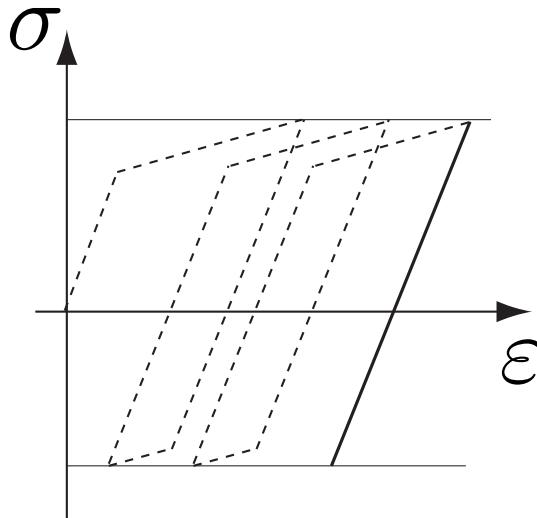


Minimum value in the wheel tread:
- 242 MPa (in compression)

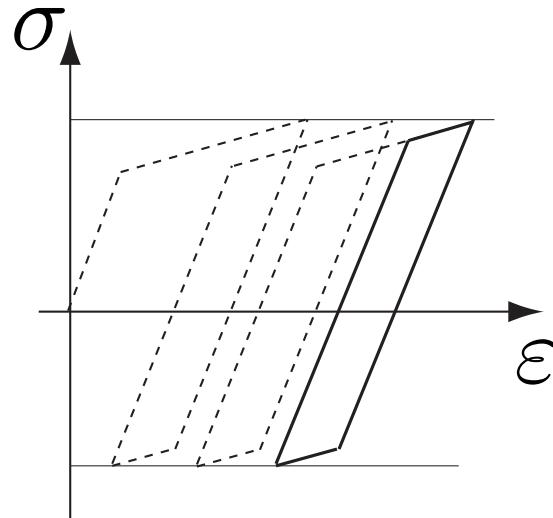


Asymptotic responses under periodic loadings

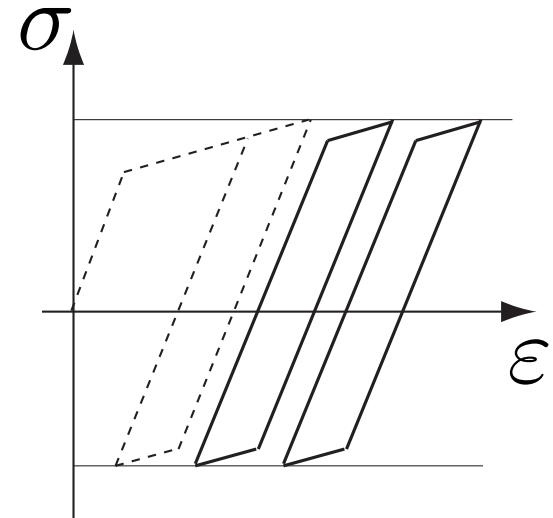
- Structures submitted to periodic loadings: what are the possible **asymptotic responses?**
 - is the structure always elastic?
 - is there an asymptotic response when $t \rightarrow \infty$
 - what is the asymptotic response when $t \rightarrow \infty$



Elastic shakedown

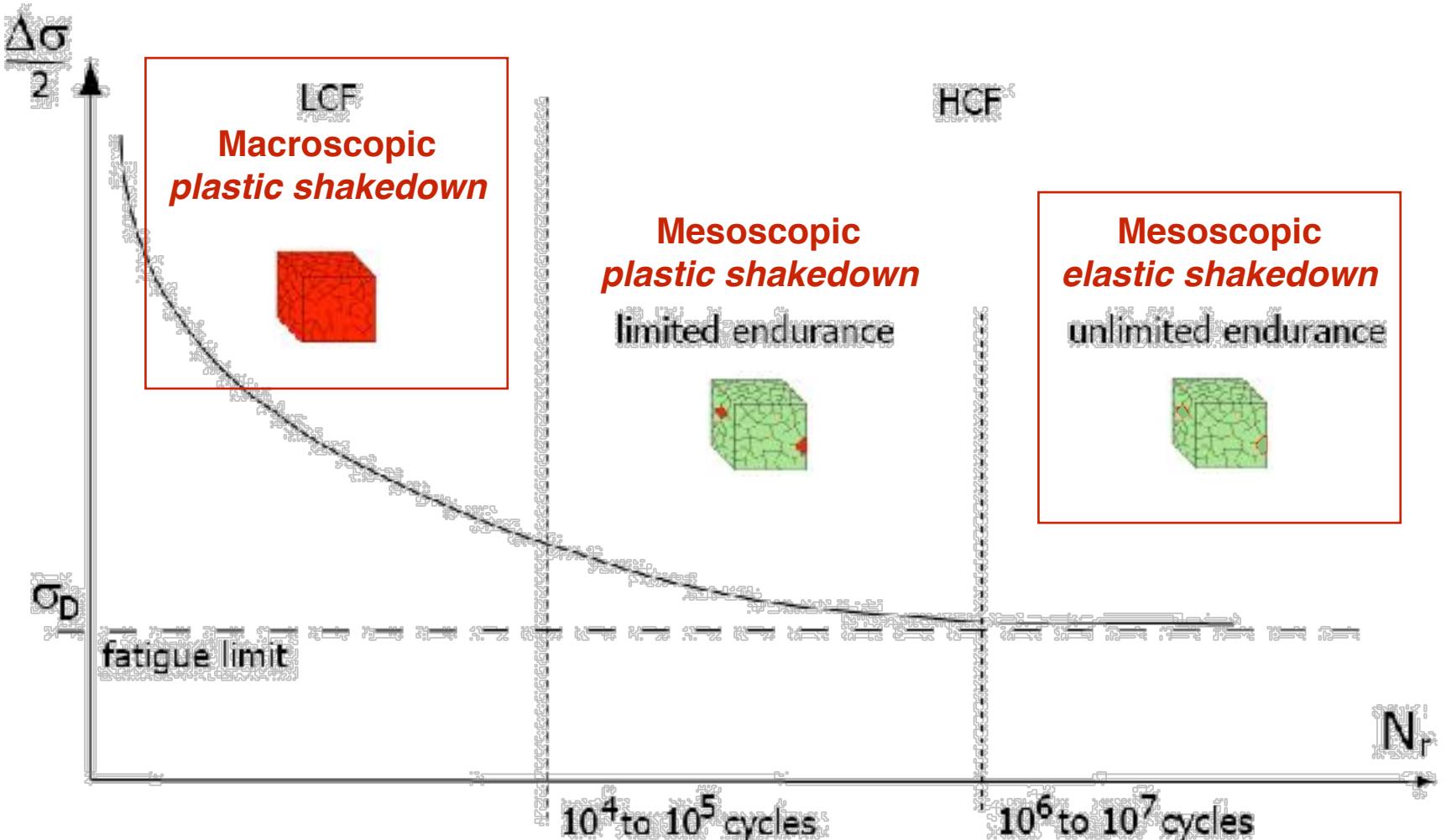


Plastic shakedown



Ratcheting

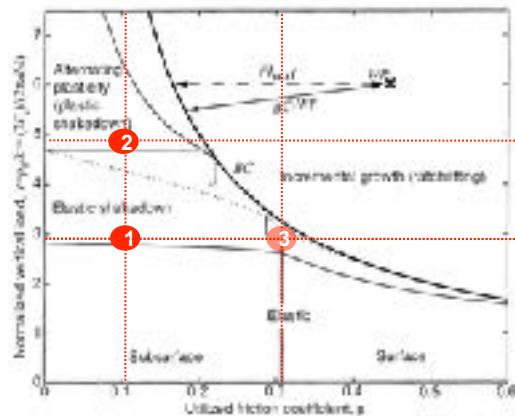
Fatigue and shakedown



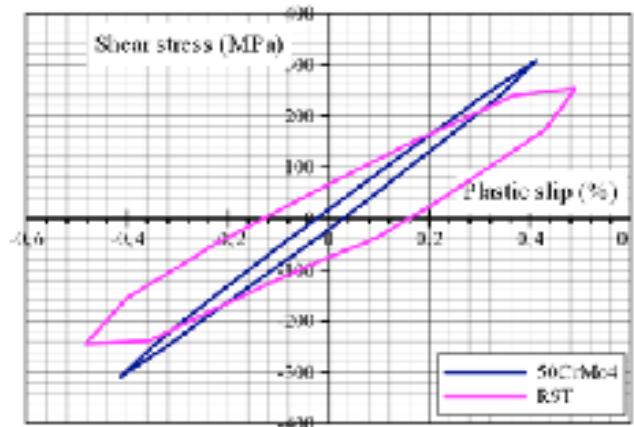
Shakedown state: steady-state algorithm

Dang Van and Maitournam (1993)

Shakedown map

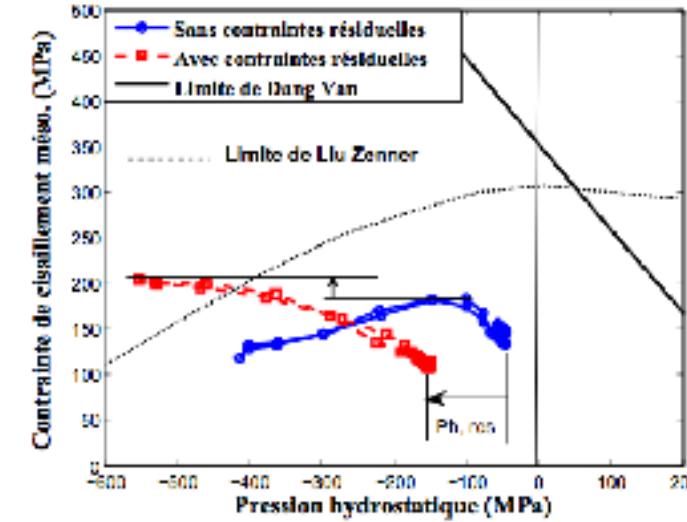


Elastoplastic simulation

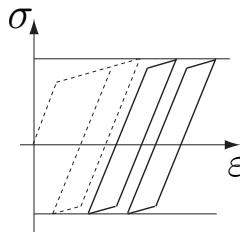
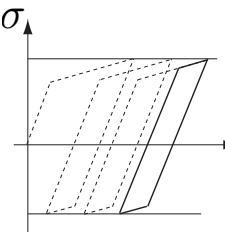
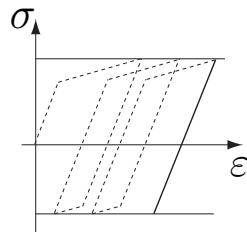


Eulerian model: $\frac{dA}{dt} = \frac{\partial A}{\partial t} + v \cdot \text{grad}(A)$

- stationary assumption
- one streamline = one cycle



Dang Van's diagram:
see next part



Proposed framework

Shakedown based framework in plasticity and in fatigue

Constitutive models?

Parameters calibration?

Fatigue criterion?
See next part

Algorithms?

Numerical implementation?
(local)

FE simulation?
(global)

Outline

- Material scale
 - **yield surface**
 - **strain hardening**
 - Structure scale
 - **structural hardening**
 - **residual stresses**
 - Numerical aspects
 - Material scale : **radial-return**
 - Structure scale : **direct algorithms**
 - Some illustrations, synthesis, remarks and references
- 
- Constitutive models?
- Parameters calibration?
- Algorithms?

Two scales

- Material scale
 - **yield surface**
 - **strain hardening**
- Structure scale
 - **structural hardening**
 - **residual stresses**
- Numerical aspects
 - Material scale : **radial-return**
 - Structure scale : **direct algorithms**
- Some illustrations, synthesis, remarks and references

Constitutive models?

Parameters calibration?

Algorithms?

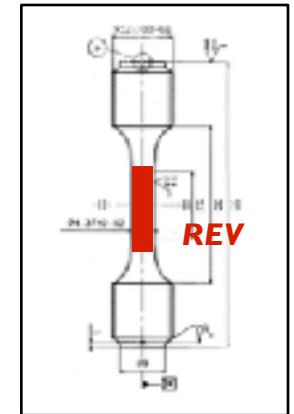
Plasticity: material scale

■ Material scale:

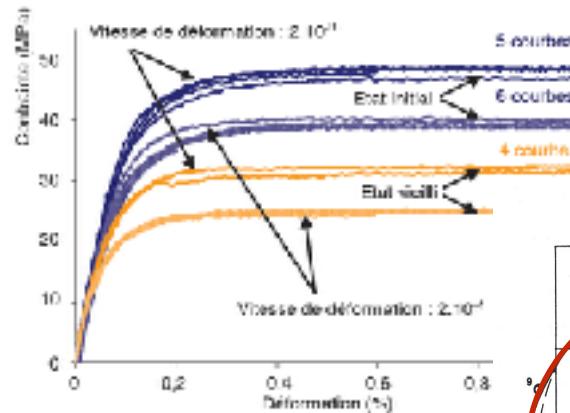
- question: **when does plasticity occur?**
 - from uniaxial to multiaxial plasticity: plastic criterion versus **yield stress**
- question: **how does plastic flow occur?**
 - flow rule: **hardening** concept

■ Structure scale:

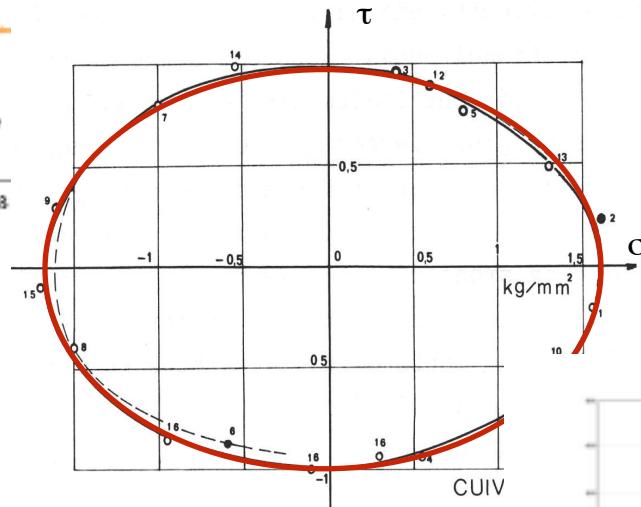
- structural hardening
 - impact of stress/strain heterogeneities
- residual stresses
 - impact of strain incompatibilities



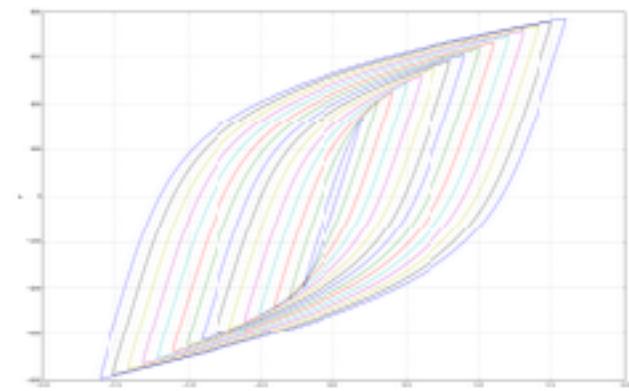
From experimental facts



Uniaxial: 1D

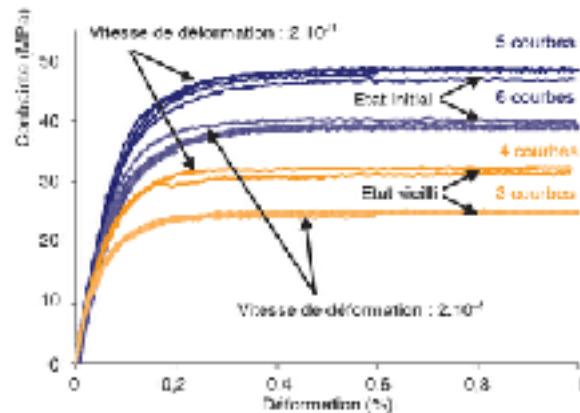


Multiaxial: 3D



Cyclic

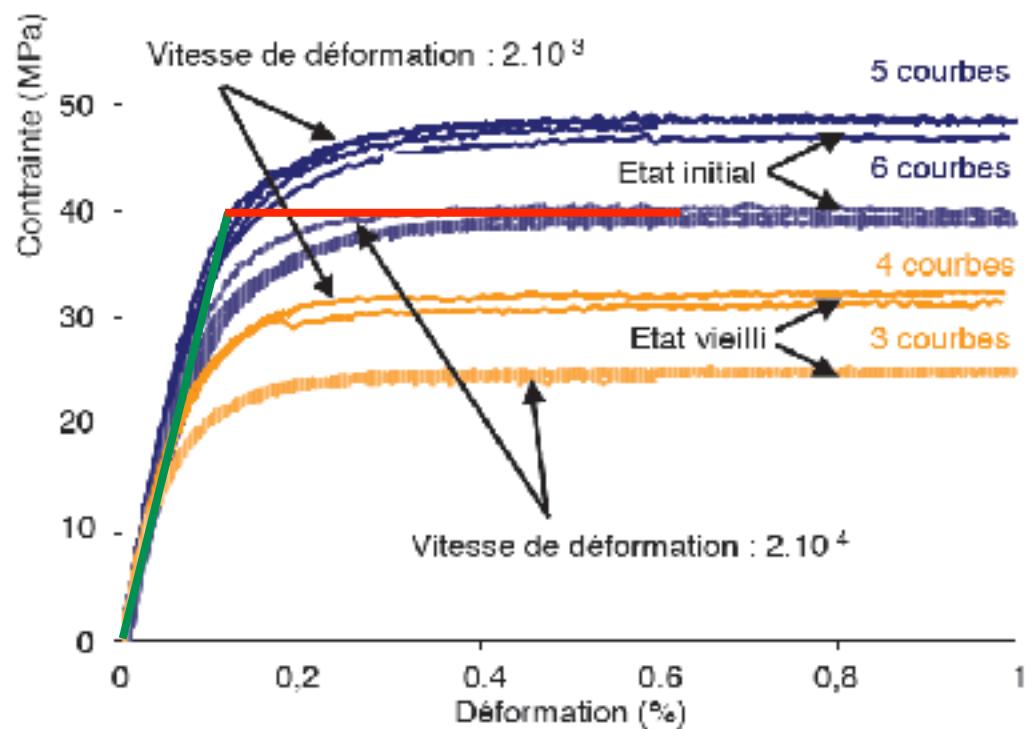
From experimental facts



Uniaxial: ID

1D plasticity

Tin-Silver-Copper alloy
 $T^{\circ}\text{C} = 20^{\circ}\text{C}$



[Dompierre et al., 2011]

ID perfect plasticity

- ▶ Perfect plasticity:

- ▶ Yield stress: $f(\sigma) = |\sigma| - \sigma_0$

- ▶ Strain additive decomposition:

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

- ▶ Elasticity:

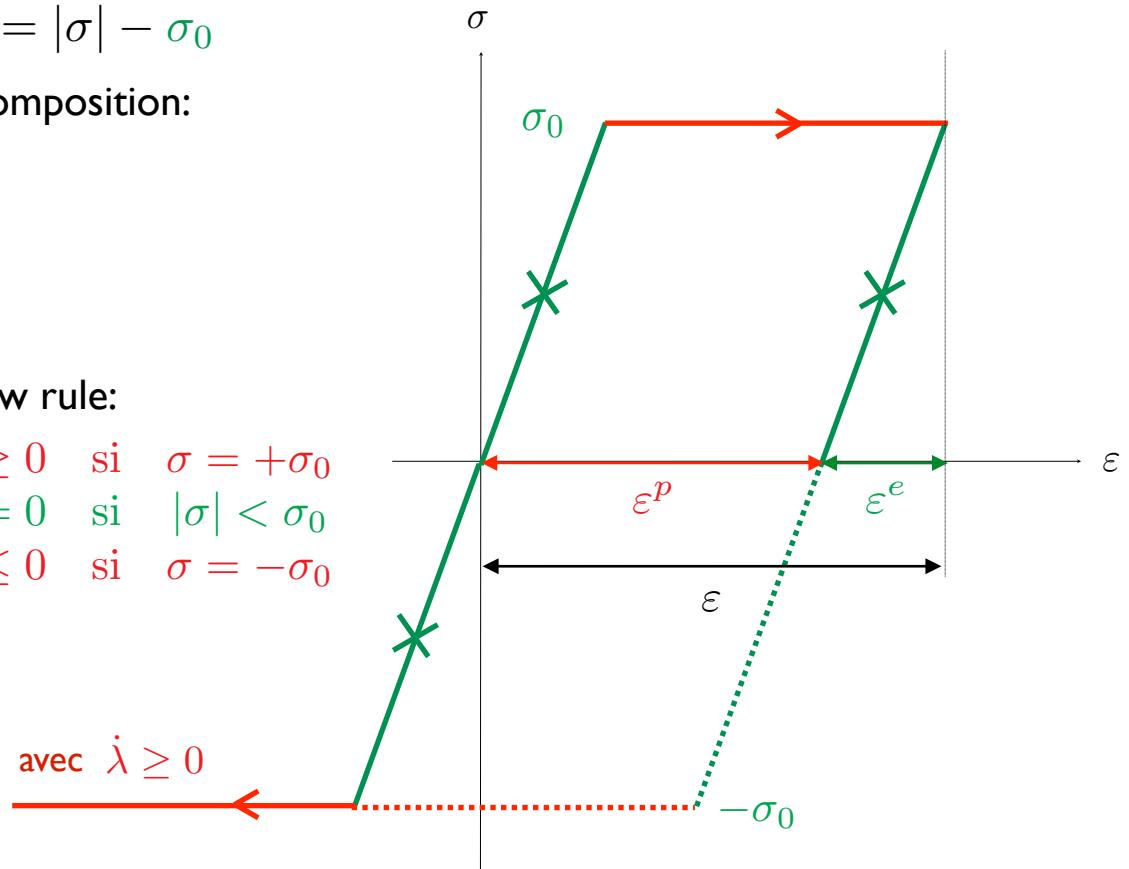
$$\sigma = E(\varepsilon - \varepsilon^p)$$

- ▶ Yield stress and flow rule:

$$|\sigma| \leq \sigma_0, \begin{cases} \dot{\varepsilon}^p \geq 0 & \text{si } \sigma = +\sigma_0 \\ \dot{\varepsilon}^p = 0 & \text{si } |\sigma| < \sigma_0 \\ \dot{\varepsilon}^p \leq 0 & \text{si } \sigma = -\sigma_0 \end{cases}$$

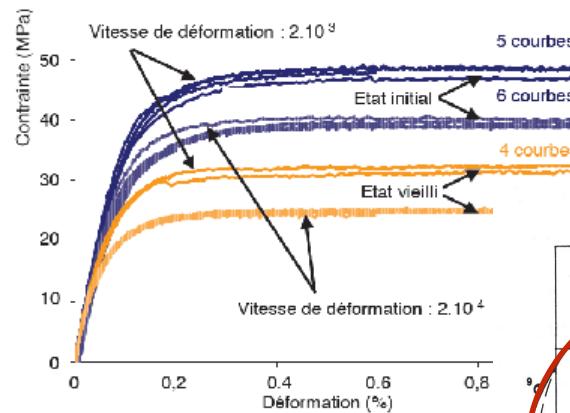
- ▶ $\dot{\varepsilon}^p$ du signe de σ_0

$$\dot{\varepsilon}^p = \dot{\lambda} \operatorname{signe}(\sigma_0) \text{ avec } \dot{\lambda} \geq 0$$

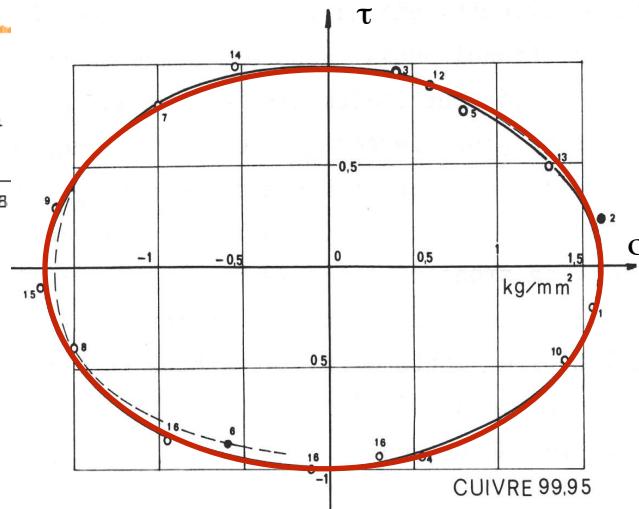


3D generalization?

From experimental facts



Uniaxial: 1D



Multiaxial: 3D

From 1D to 3D plasticity

- ▶ **What about a generalization ?**
 - ▶ from yield stress to **plastic/yield criterion** and/or **yield surface** :

$$f(\underline{\sigma}) = |\sigma| - \sigma_0 \longrightarrow f(\underline{\sigma}) \leq 0$$

- ▶ History: multiaxial test in **tension-torsion**



G.I Taylor
(Cavendish lab., Cambridge)



H-D. Bui
(LMS, X)



S. Calloch
(IRDL, ENSTA Brest)



V.Aubin
(MSSMAT, Centrale Paris)

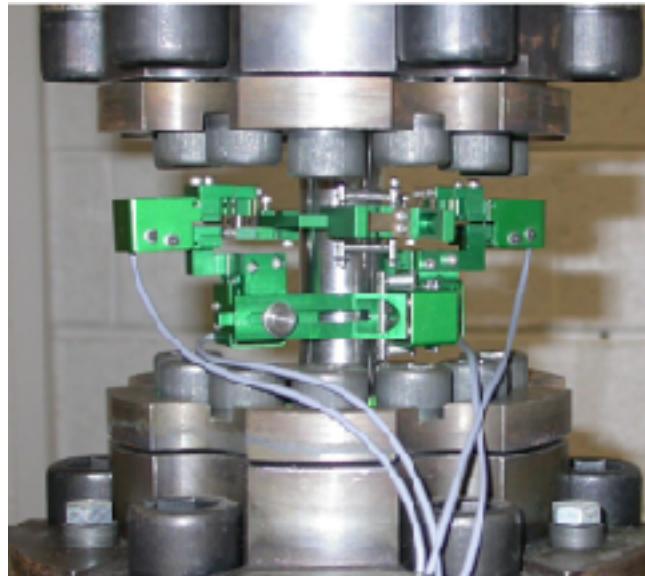
From 1D to 3D plasticity: yield stress

- ▶ **What about a generalization ?**

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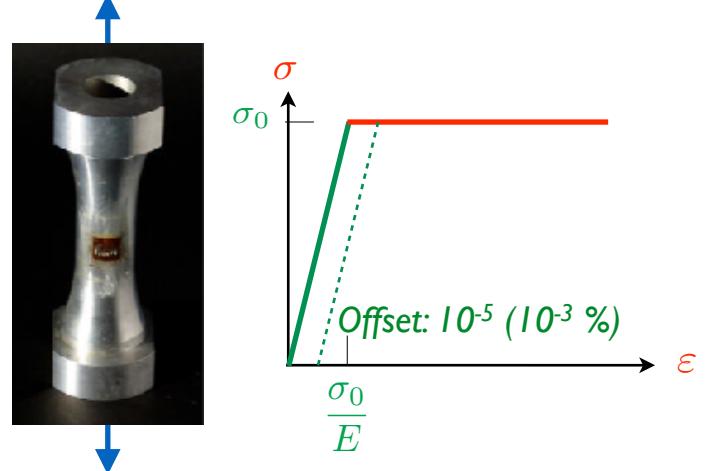
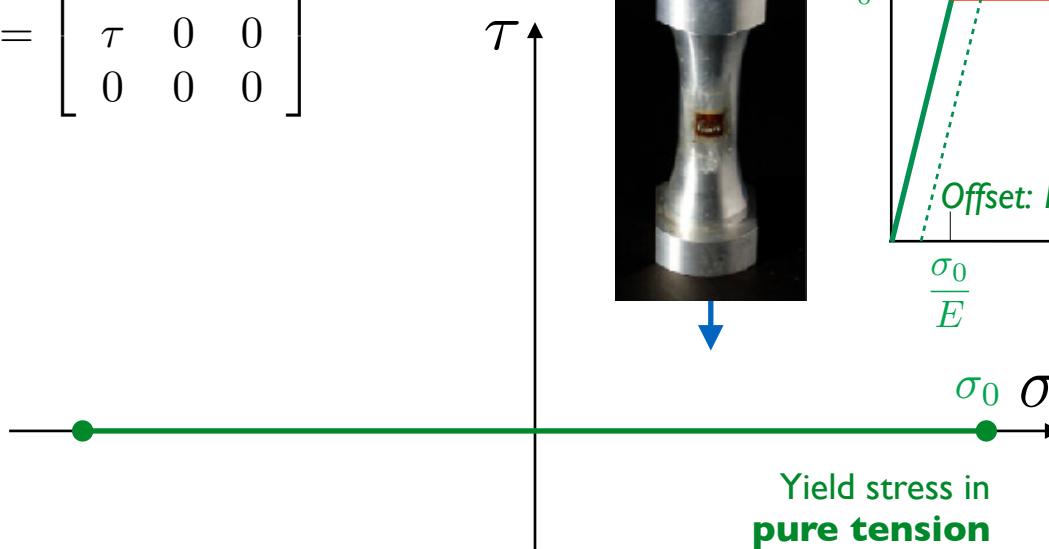


$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Tension-torsion tests

- ▶ Yield stress determination in pure tension

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Tension-torsion tests

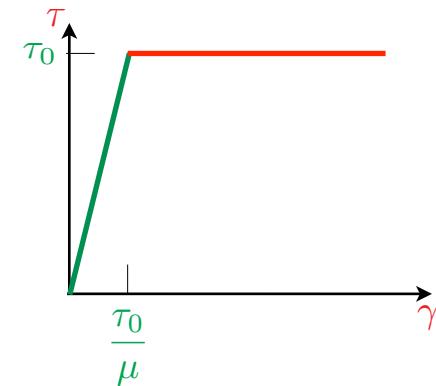
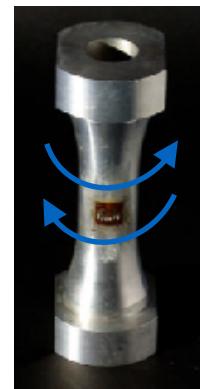
► Yield stress determination in pure torsion

$$\underline{\sigma} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Limite d'élasticité
en **torsion pure**

$$\tau \uparrow$$

 τ_0



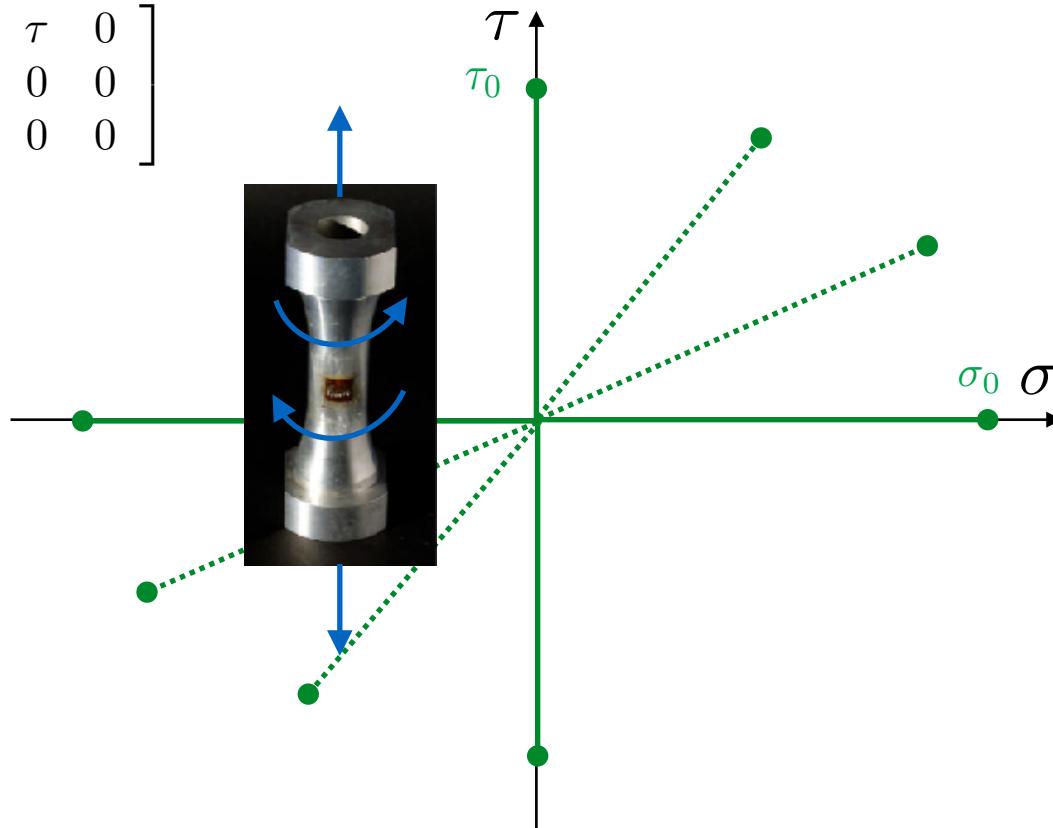
$$\sigma_0 \quad \sigma$$

Yield stress in
pure tension

Tension-torsion tests

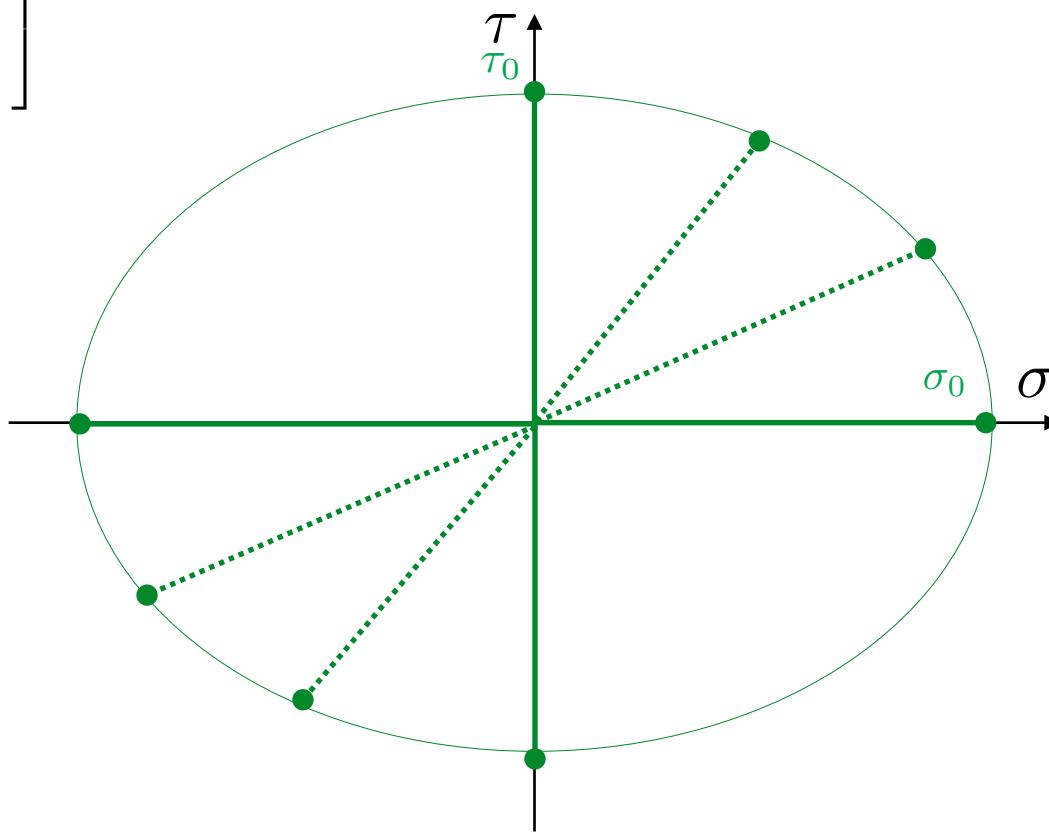
- ▶ Yield stress determination for radial loadings

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

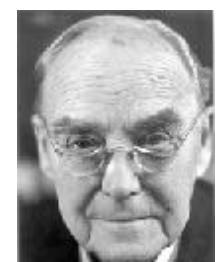
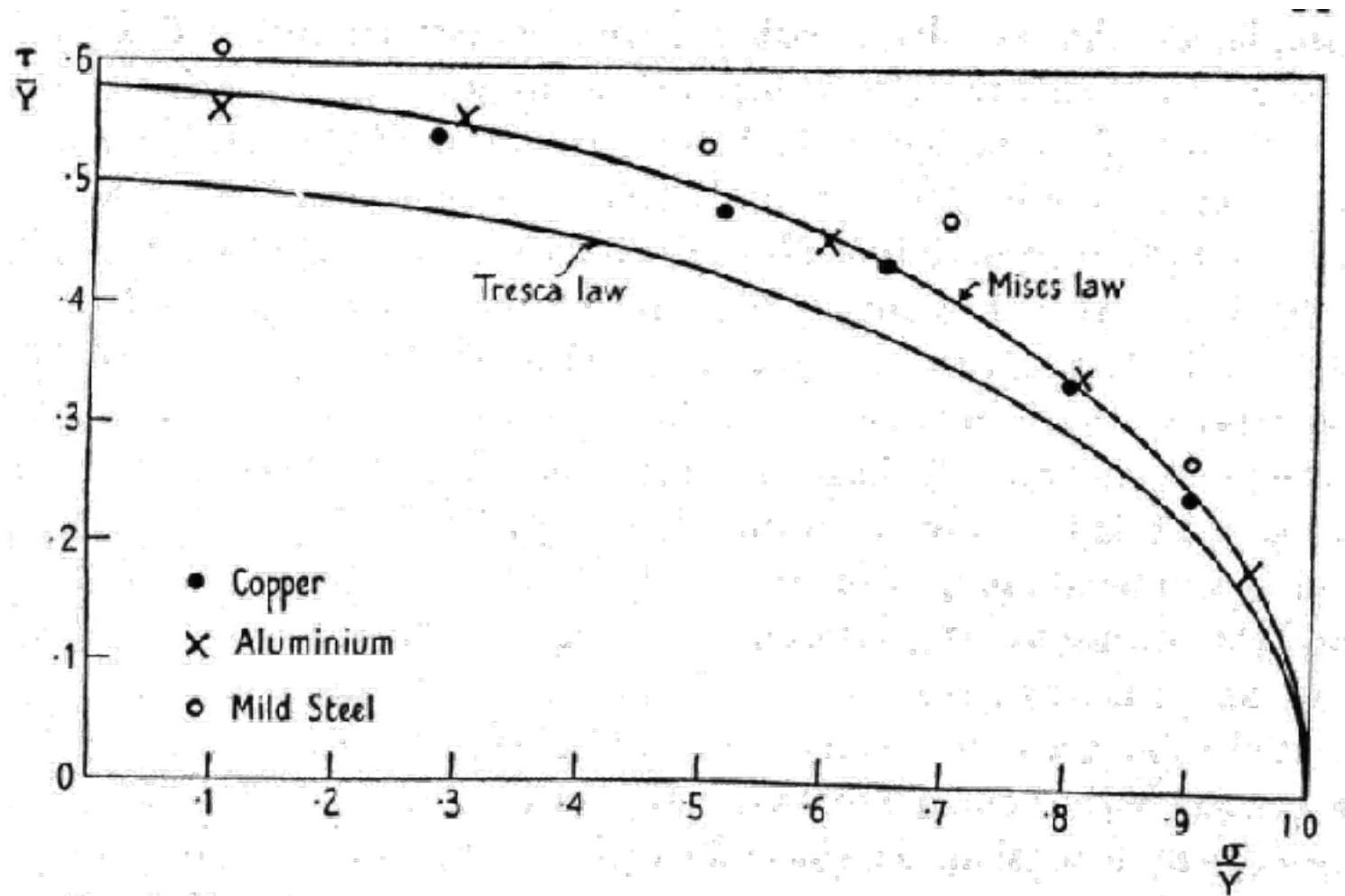


Yield surface in tension-torsion

$$\underline{\sigma} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

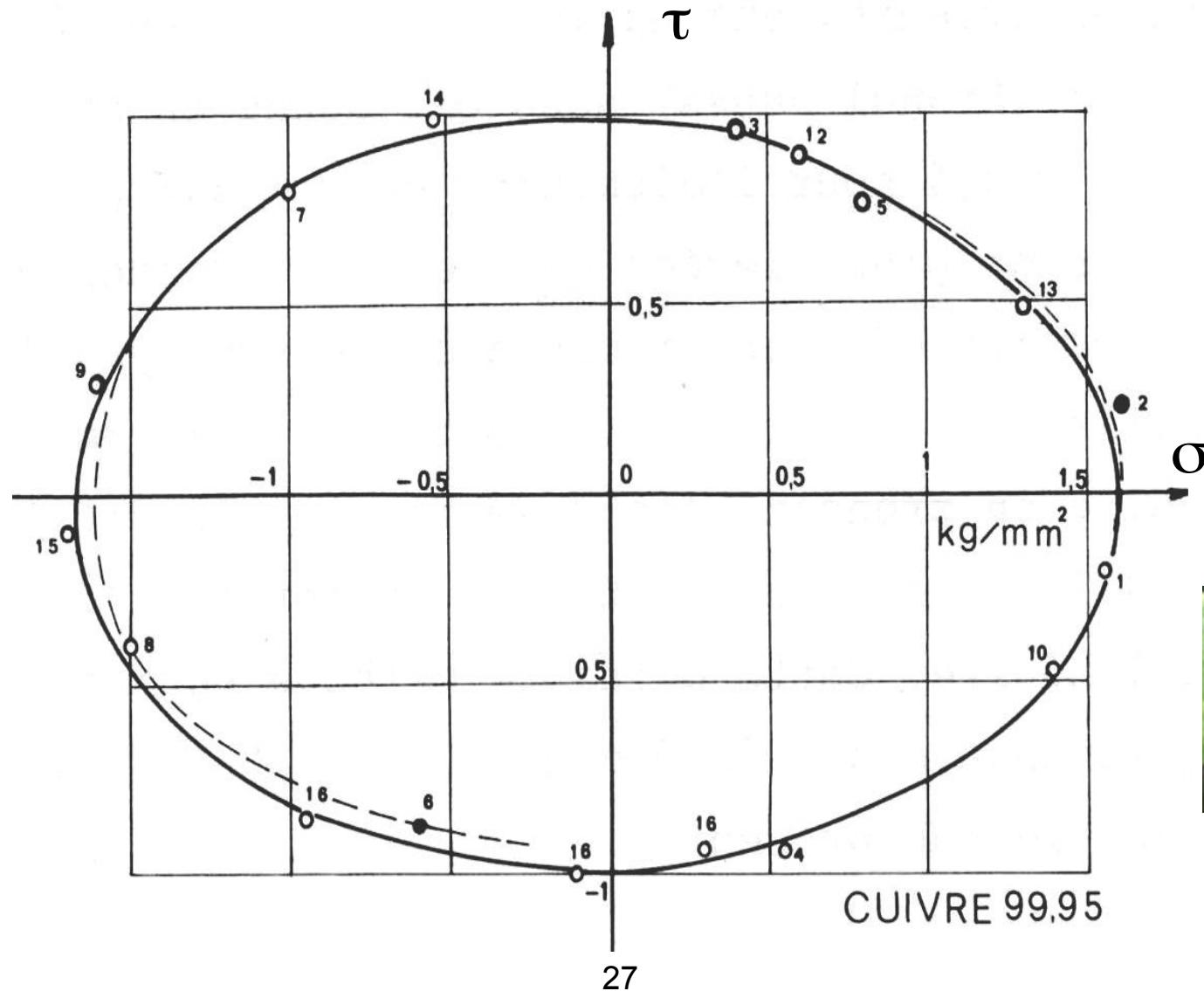


Taylor's experiments (1931)



G.I. Taylor
(1886-1975)

Bui's experiments (1969)

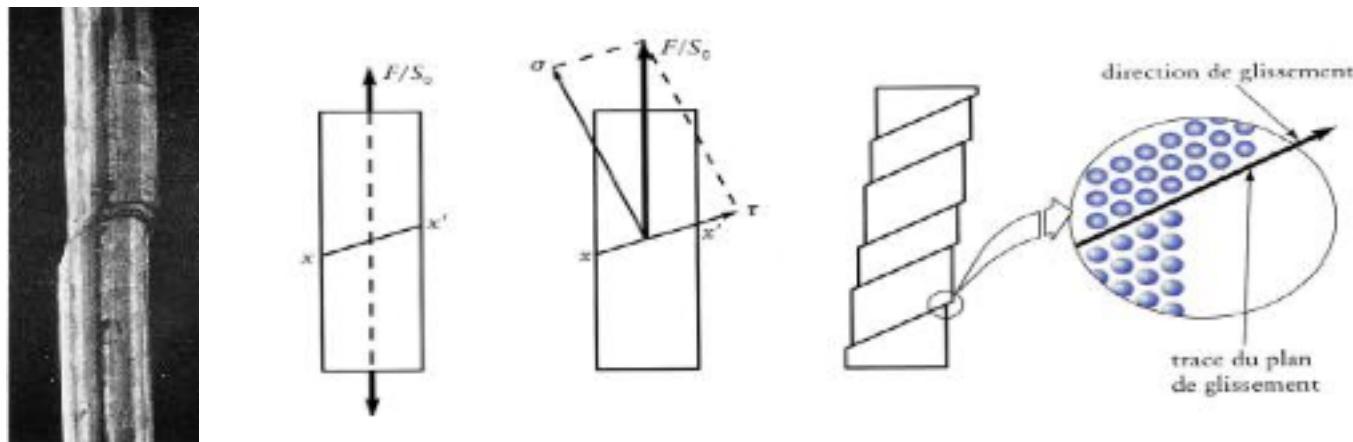


H-D. Bui
(1937-2013)

[Bui, 1969]

3D plasticity: yield stress

- ▶ physical mechanisms: **plastic slip** (cristal planes, dislocations)



H.Tresca
(1814-1885)

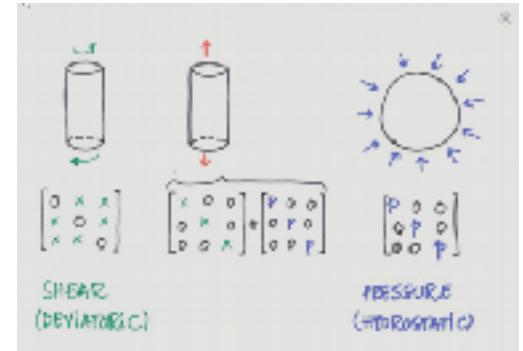
- ▶ plastic deformation **without volume change**
- ▶ activated with **shear stress**
- ▶ **What about yield criteria:** two main theories
 - ▶ criterion based on **maximal shear stress** (Tresca, 1874)
 - ▶ criterion based on **maximum distortion energy** (von Mises, 1913)
 - ▶ **no influence of the hydrostatic pressure** (*no volume change*)

von Mises plastic criterion

- ▶ **Maximum distortion energy criterion:**

- ▶ based on **deviatoric stress**,

$$\underline{\sigma} = \underline{s} + \frac{1}{3} \operatorname{tr}(\underline{\sigma})$$



- ▶ isotropic function of the stress tensor = function of its **invariants**
- ▶ function of the **second invariant** of the deviator:

$$J_2 = \frac{1}{2} \underline{s} : \underline{s}$$

- ▶ **von Mises criterion [1913]**

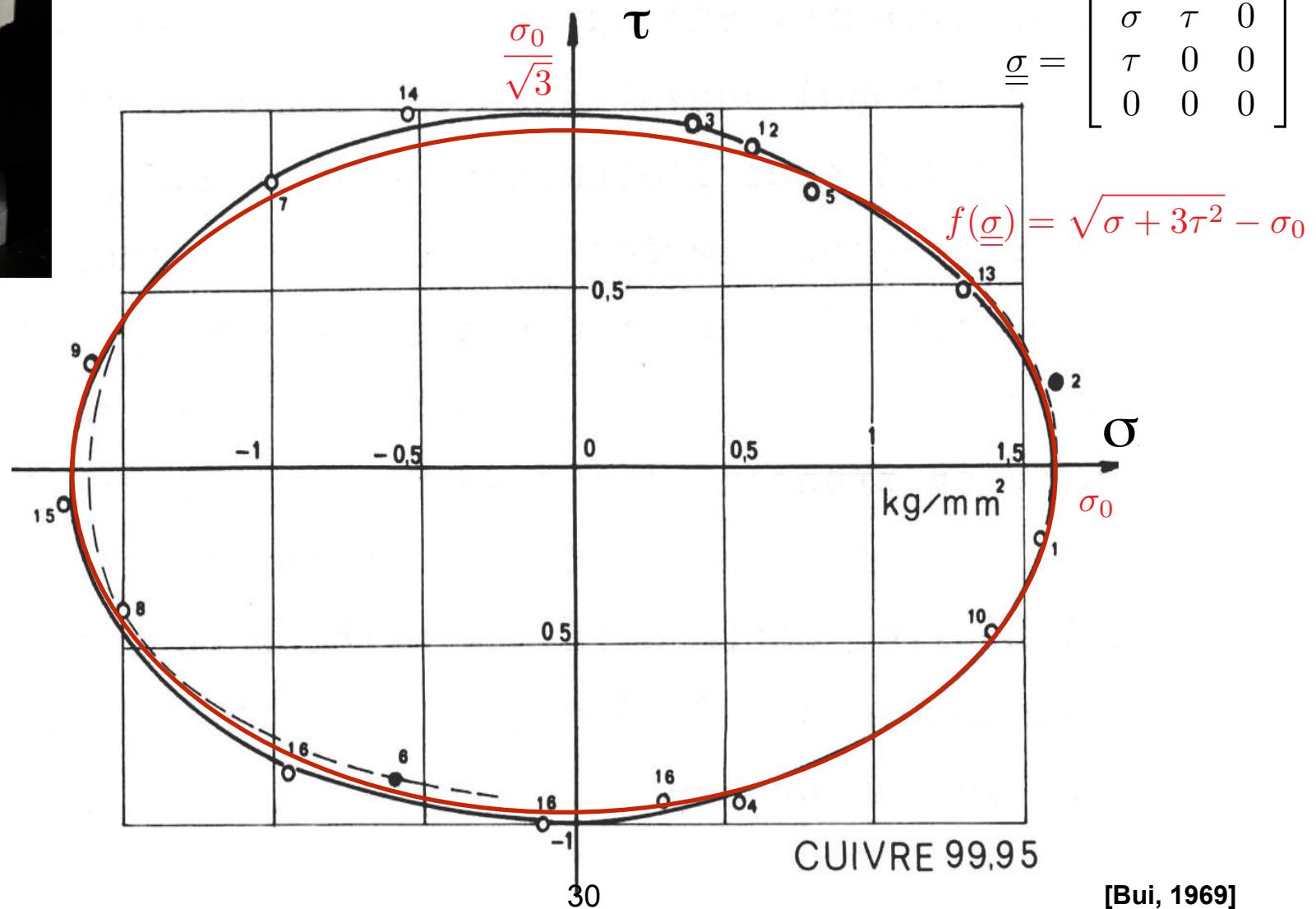
$$f(\underline{\sigma}) = \sqrt{\frac{3}{2} \underline{s} : \underline{s}} - \sigma_0$$

$$f(\underline{\sigma}) = \sqrt{\frac{3}{2} \underline{\sigma} : \underline{\sigma} - \frac{1}{2} \operatorname{tr}(\underline{\sigma})^2} - \sigma_0$$



R. von Mises
(1883-1953)

von Mises in tension-torsion



From 1D to 3D plasticity: flow rule

- ▶ **3D generalization ?**

- ▶ Total strain additive decomposition : $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^e + \underline{\underline{\varepsilon}}^p$
- ▶ plastic deformation **without volume change** :

$$tr(\underline{\underline{\varepsilon}}^p) = 0$$

- ▶ Yield surface :

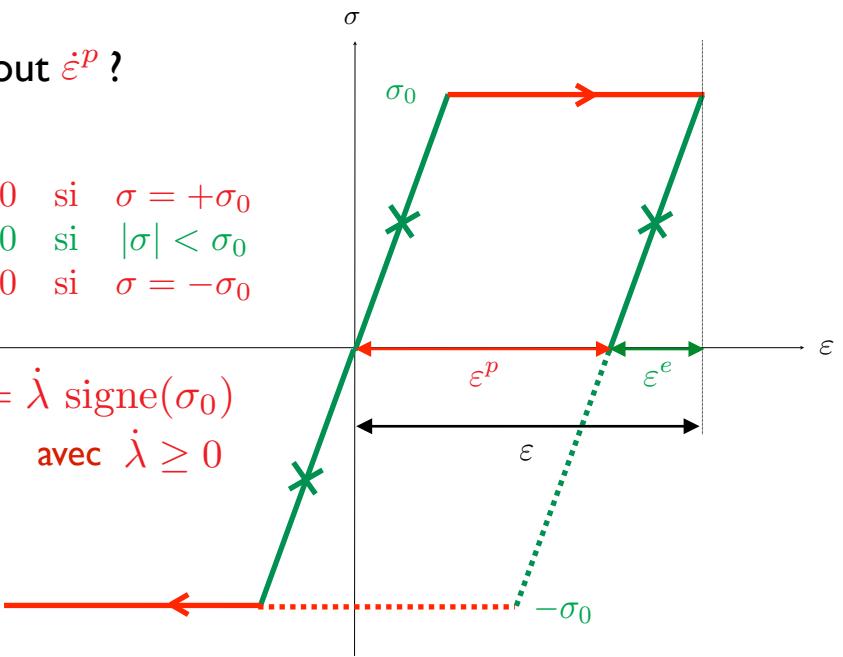
$$f(\underline{\sigma}) \leq 0$$

- ▶ And when $f(\underline{\sigma}) = 0$? What about $\dot{\varepsilon}^p$?

uniaxial $|\sigma| \leq \sigma_0, \begin{cases} \dot{\varepsilon}^p \geq 0 & \text{si } \sigma = +\sigma_0 \\ \dot{\varepsilon}^p = 0 & \text{si } |\sigma| < \sigma_0 \\ \dot{\varepsilon}^p \leq 0 & \text{si } \sigma = -\sigma_0 \end{cases}$

$$\dot{\varepsilon}^p = \dot{\lambda} \operatorname{signe}(\sigma_0)$$

avec $\dot{\lambda} \geq 0$



Tension-torsion tests: strain measurements

- ▶ Strain gauges :

$$\underline{\underline{\varepsilon}}$$

- ▶ Applied force and torque :

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\varepsilon}}^e = \underline{\underline{C}}^{-1} : \underline{\underline{\sigma}}$$

- ▶ Plastic strain deduced from total and elastic strain tensors:

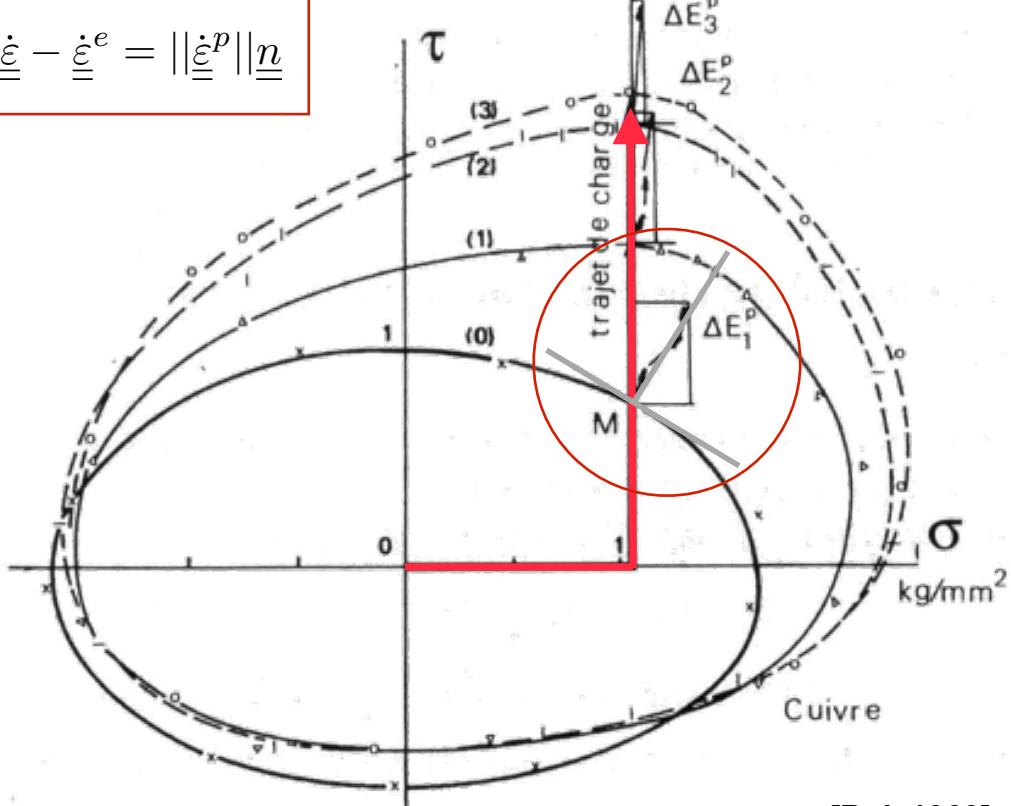
$$\underline{\underline{\varepsilon}}^p = \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^e$$

$$\dot{\underline{\underline{\varepsilon}}}^p = \dot{\underline{\underline{\varepsilon}}} - \dot{\underline{\underline{\varepsilon}}}^e = ||\dot{\underline{\underline{\varepsilon}}}^p|| \underline{\underline{n}}$$



Tension-torsion tests: normality law

$$\dot{\varepsilon}^p = \dot{\varepsilon} - \dot{\varepsilon}^e = ||\dot{\varepsilon}^p||n$$



[Bui, 1969]

3D plasticity: flow rules

► Generalization ?

- ▶ Strain additive decomposition: $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^e + \underline{\underline{\varepsilon}}^p$
- ▶ Experimental observation: plastic deformation **without volume change**

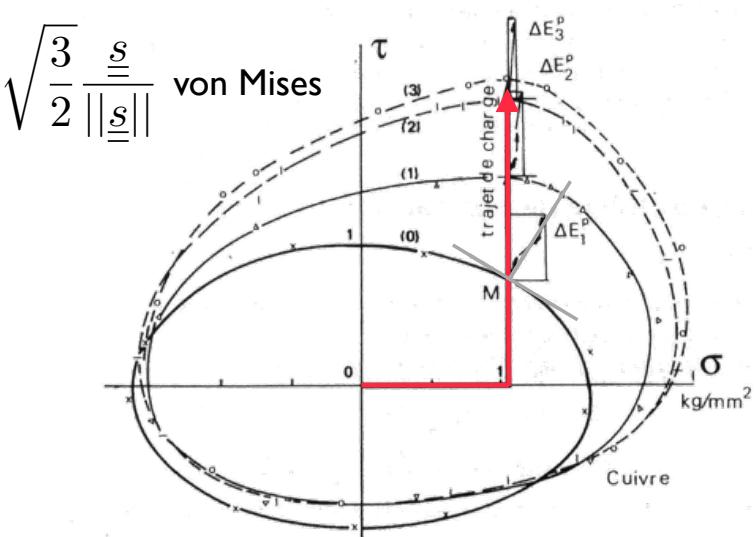
$$\text{tr}(\underline{\underline{\varepsilon}}^p) = 0$$

- ▶ Yield surface: $f(\underline{\sigma}) \leq 0$
- ▶ Flow normal to the yield surface: **collinear to the gradient** of f (normality rule)

$$f(\underline{\sigma}) = |\underline{\sigma}| - \sigma_0$$

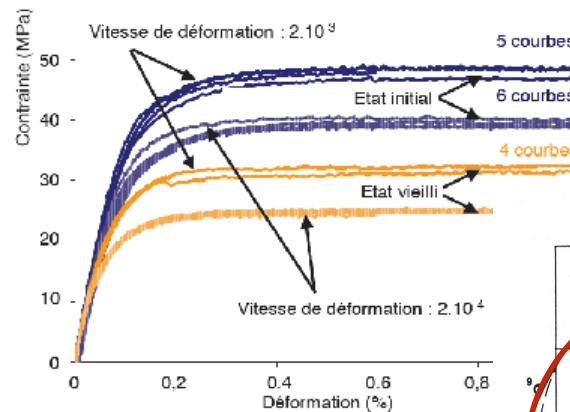
$$\dot{\underline{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \underline{\sigma}}$$

$$\dot{\underline{\varepsilon}}^p = \dot{\lambda} \sqrt{\frac{3}{2} \frac{\underline{\underline{s}}}{||\underline{\underline{s}}||}}$$

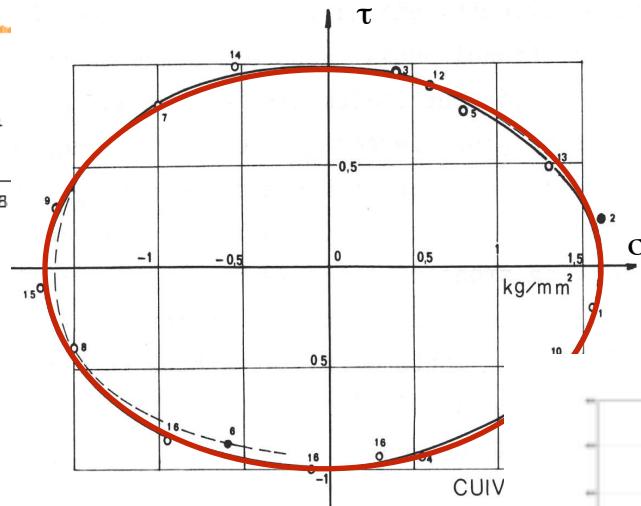


- ▶ Plastic multiplier: $\dot{\lambda} \geq 0$
- ▶ Compact equation: $\dot{\lambda} f = 0$

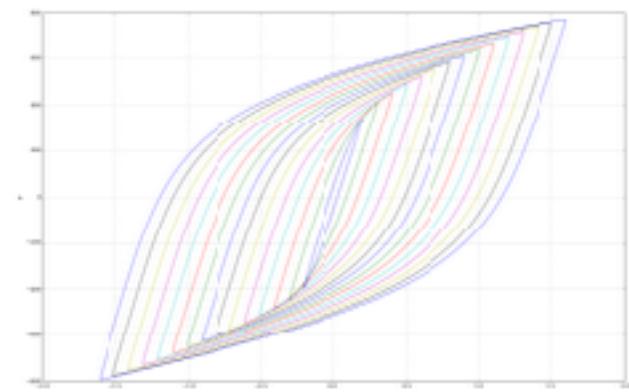
From experimental facts



Uniaxial: 1D



Multiaxial: 3D



Cyclic

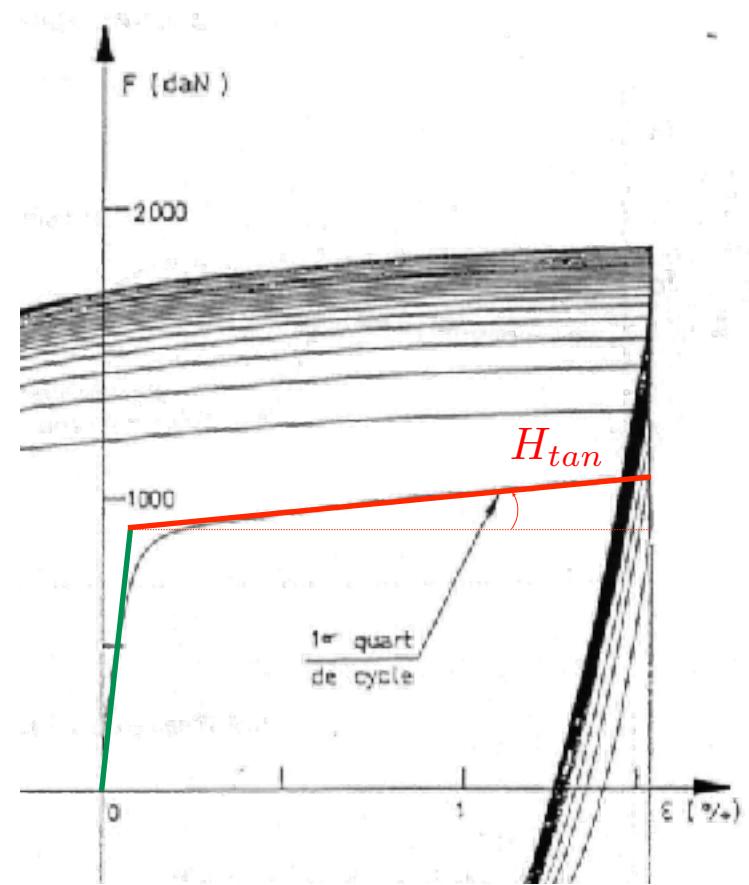
1D plasticity: hardening

- ▶ What is **hardening** ?

- ▶ Deformability evolution of the material
- ▶ **Strengthening of a metal** by plastic deformation : increasing stress is required to produce additional plastic strain

- ▶ Yield stress evolution

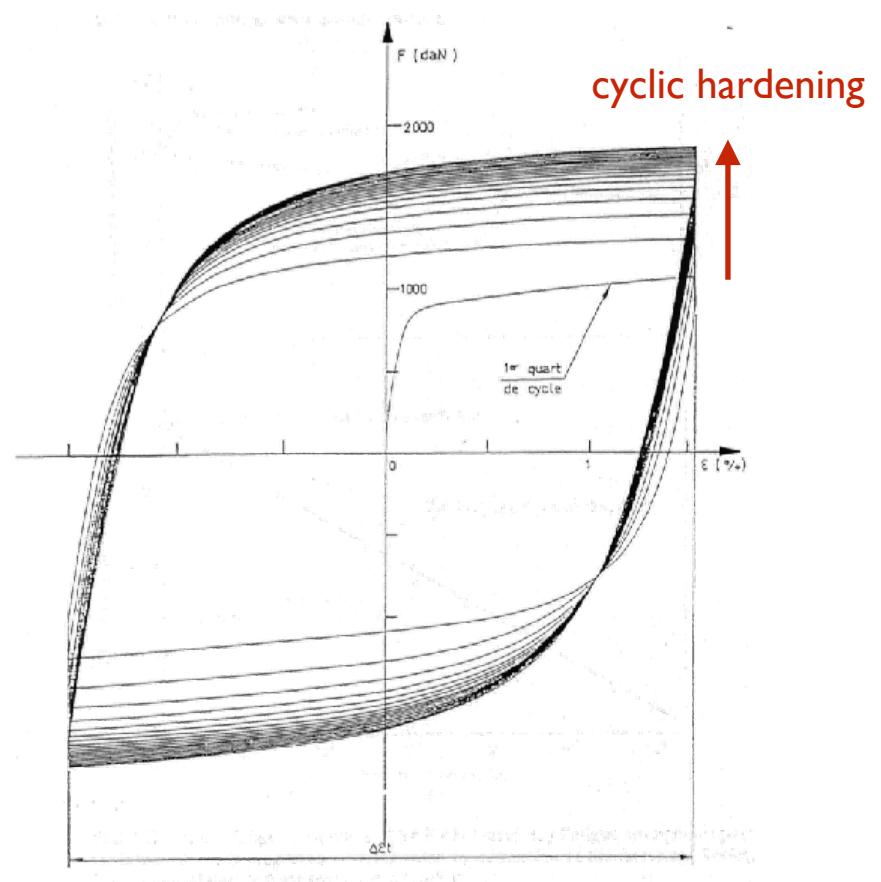
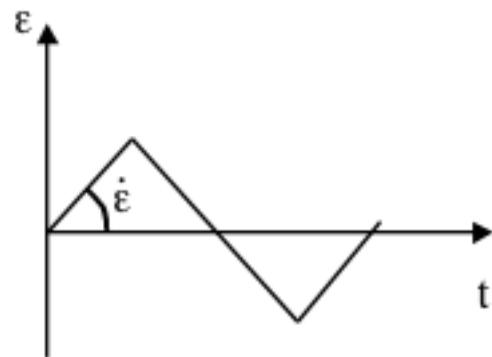
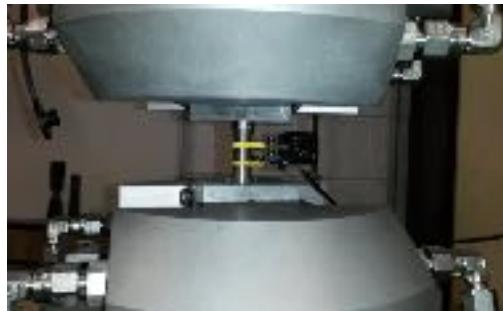
- ▶ Mechanisms : dislocation movements, dislocation density evolution as barriers to plastic slip, ...



What is **cyclic** hardening ?

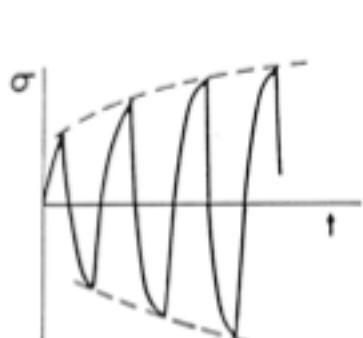
Cyclic strain-controlled tests

- **Principle:** tension-compression with increasing strain amplitude

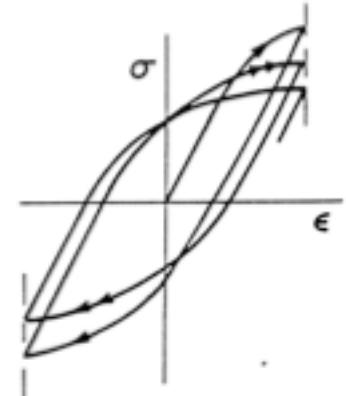
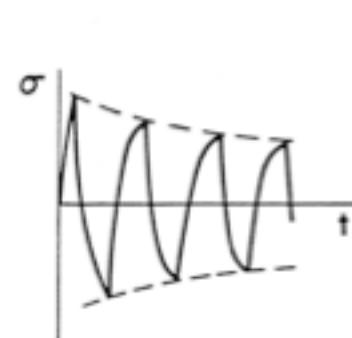
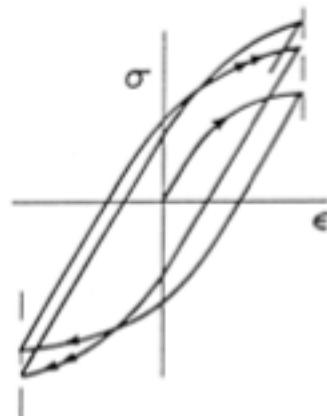


1D plasticity: cyclic hardening

- Cyclic hardening:

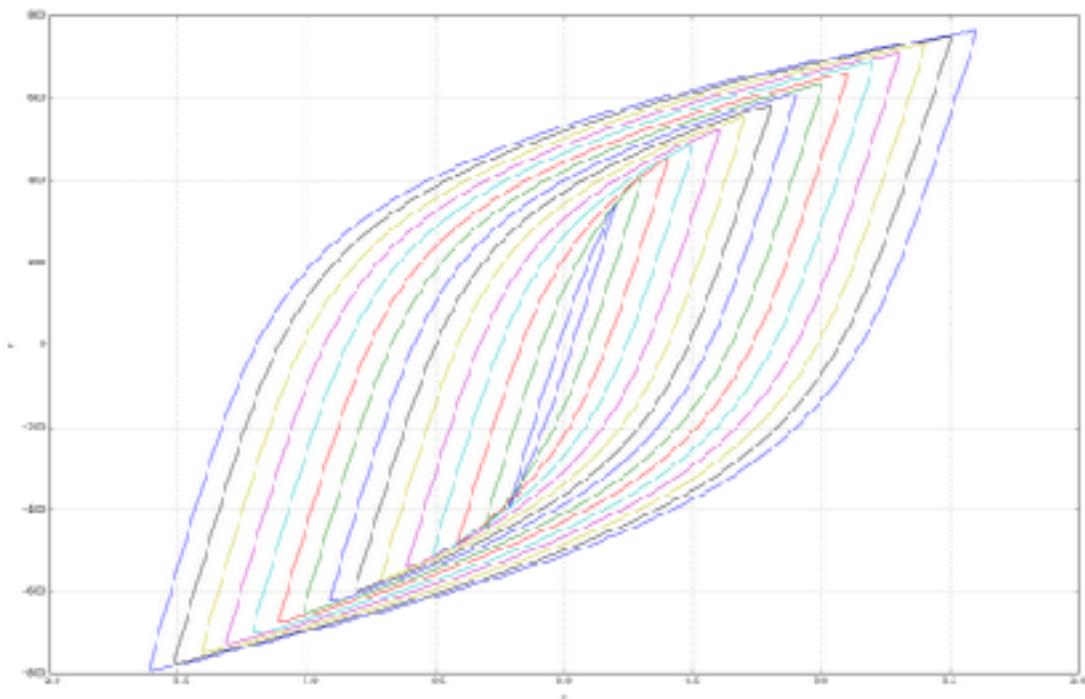
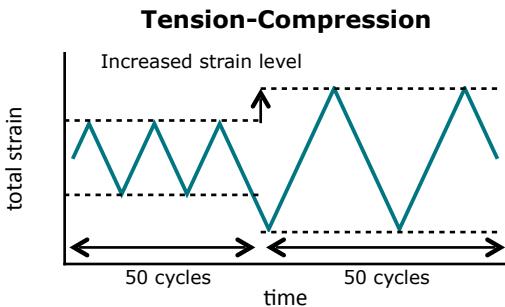
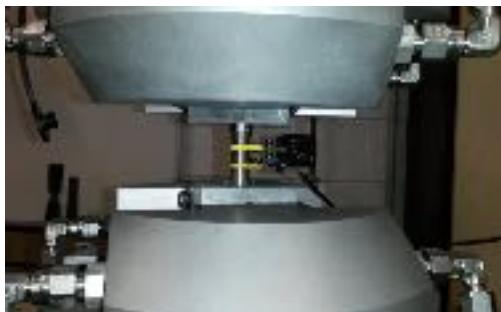


- Cyclic softening:



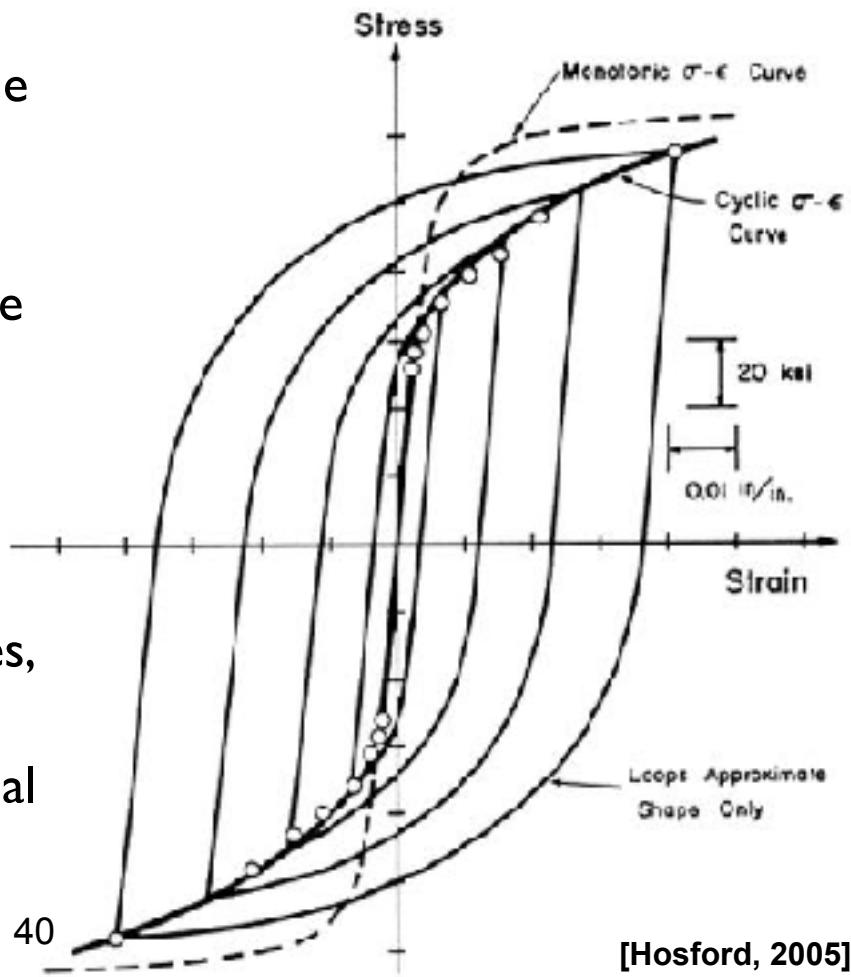
Cyclic hardening curve

- ▶ **Principle:** tension-compression with increasing strain amplitude
 - ▶ **cyclic hardening** tests
 - ▶ **strong assumption:** search for the **stabilized response** of the material in order to **calibrate the parameters** (*remember: shakedown*)

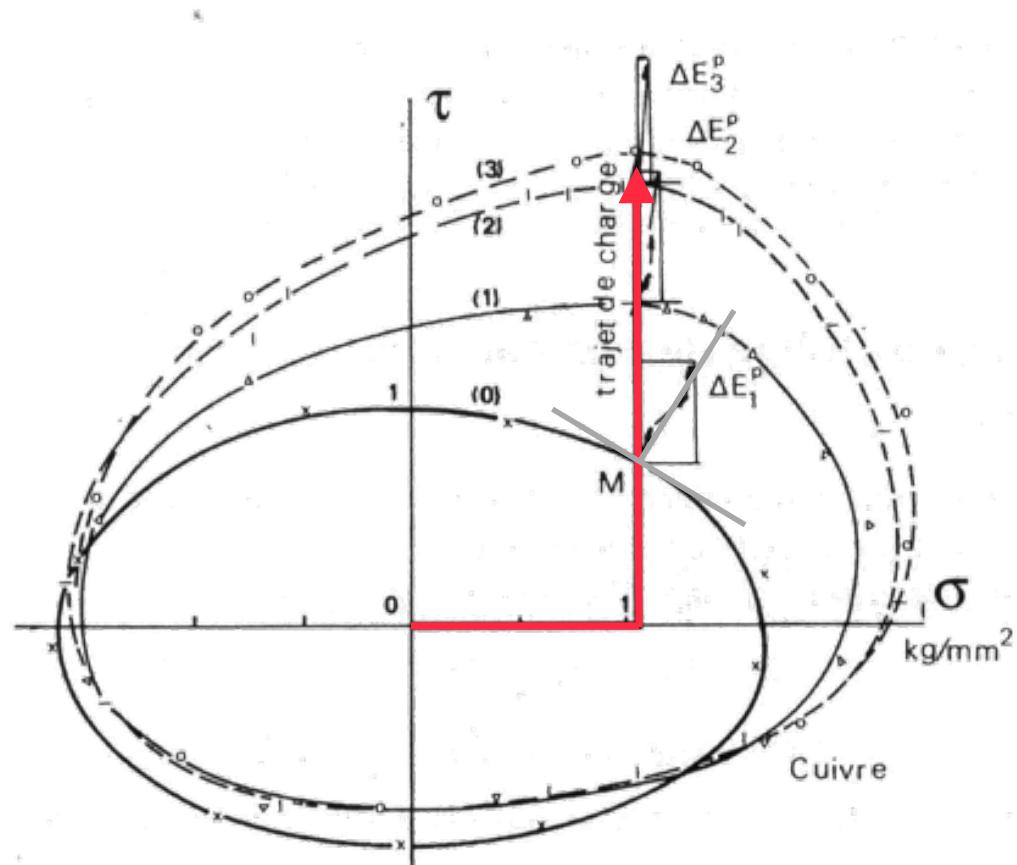


Cyclic hardening curve

- Cyclic **softening**:
 - cyclic hardening curve below the monotonic one (see figure)
- Cyclic **hardening**:
 - cyclic hardening curve above the monotonic one
- Depends on:
 - hardening **mechanisms**
(dislocations density, precipitates, metallic inclusions, ...)
 - manufacturing **process**, material residual state, ...



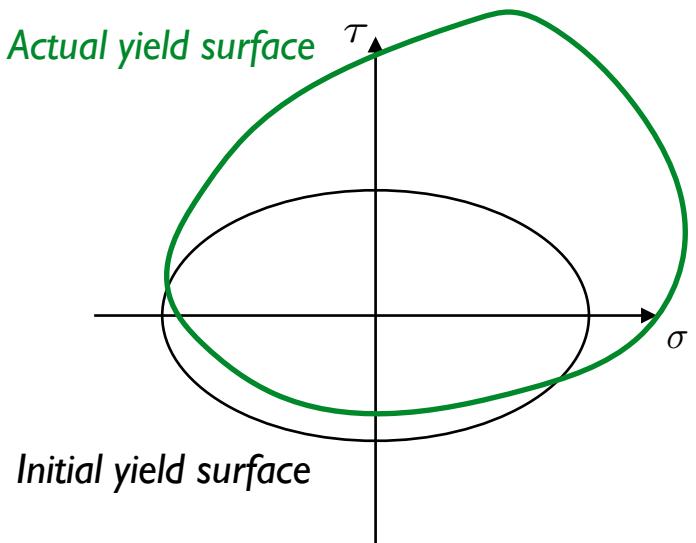
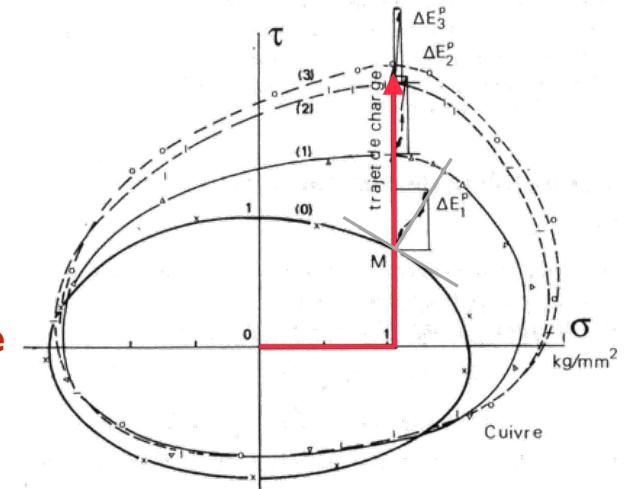
3D plasticity: hardening



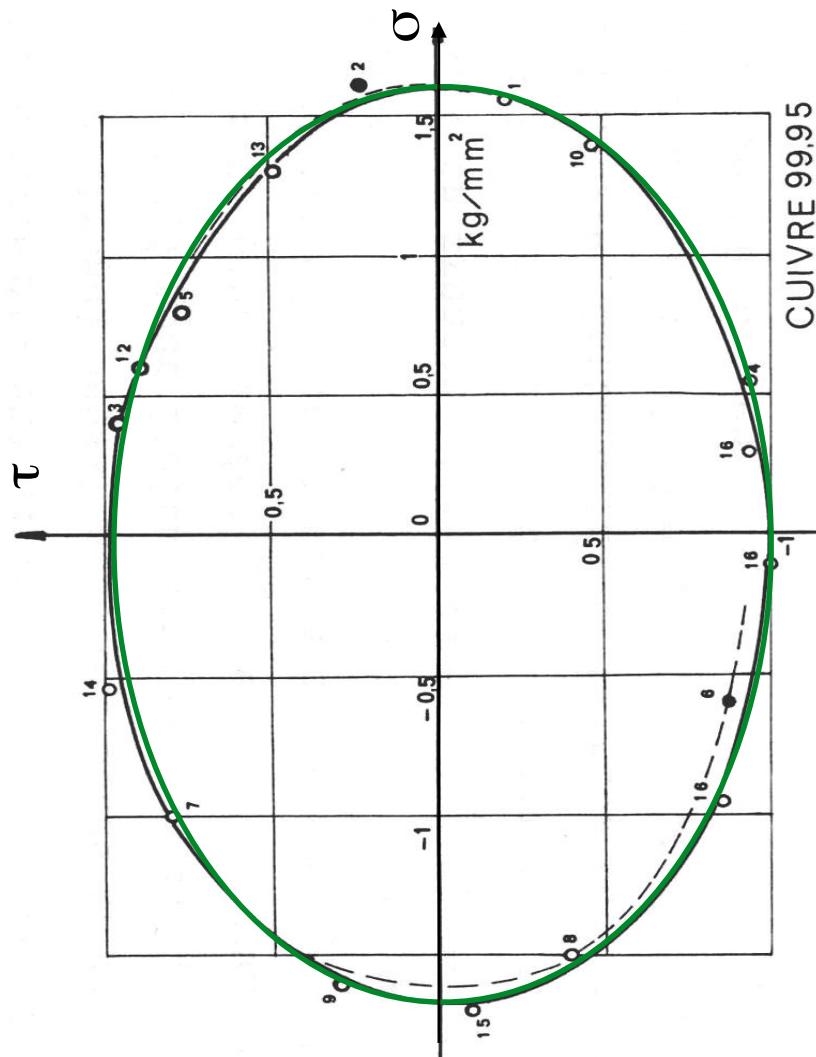
3D plasticity: hardening

- ▶ How to generalize ?

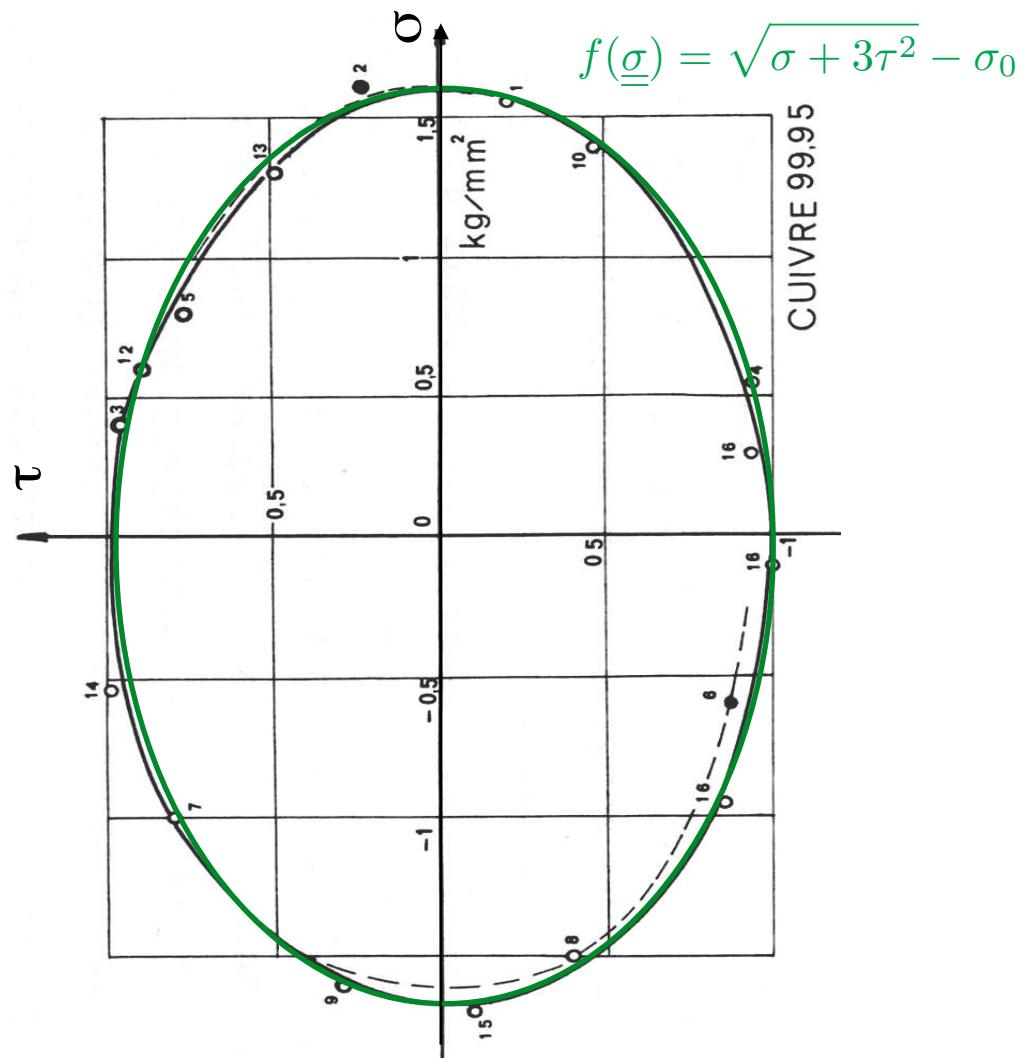
- ▶ Evolution of the **yield surface**
- ▶ Size variations: **isotropic case**
- ▶ Movements in the stress space: **kinematic case**
- ▶ Distortion: delicate modeling problem (see *ratcheting*)
- ▶ Hardening
 - ▶ Isotropic case: **scalar**
 - ▶ Kinematic case: **tensor**

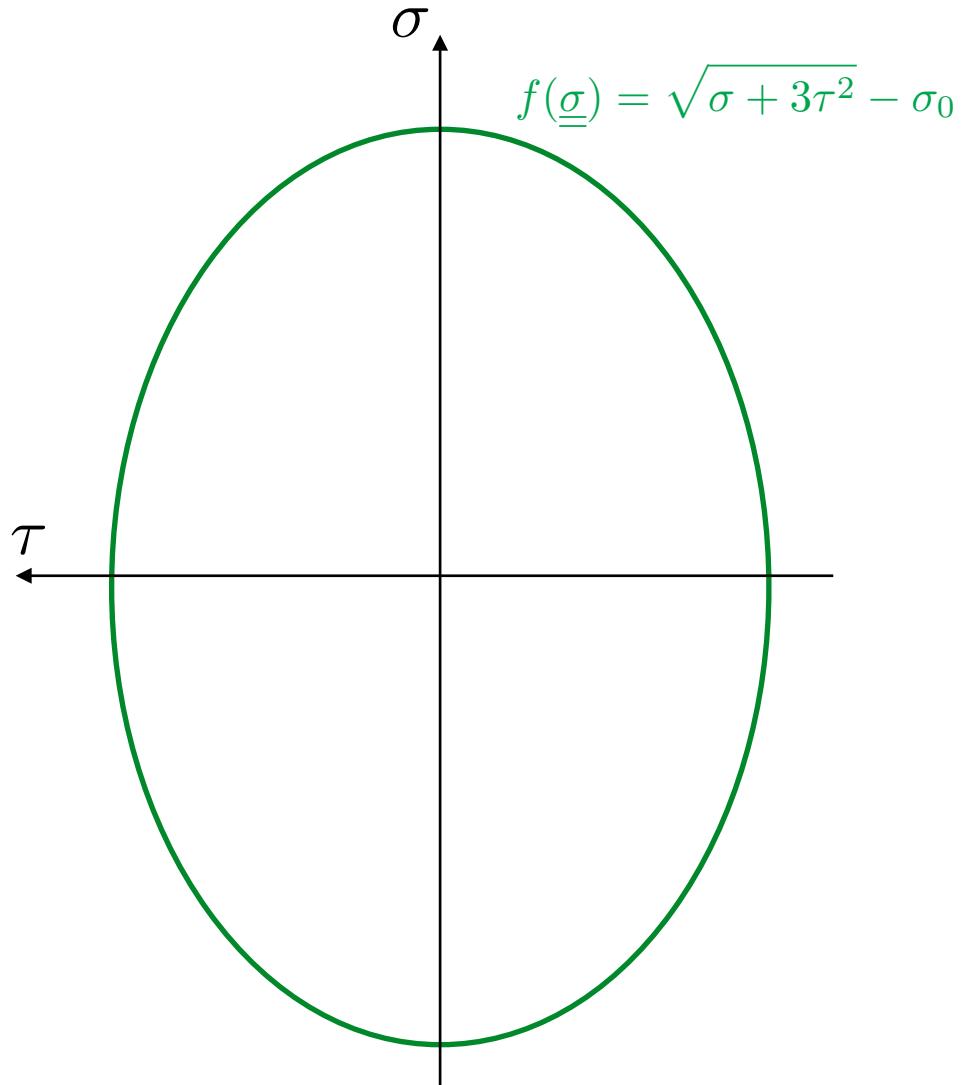


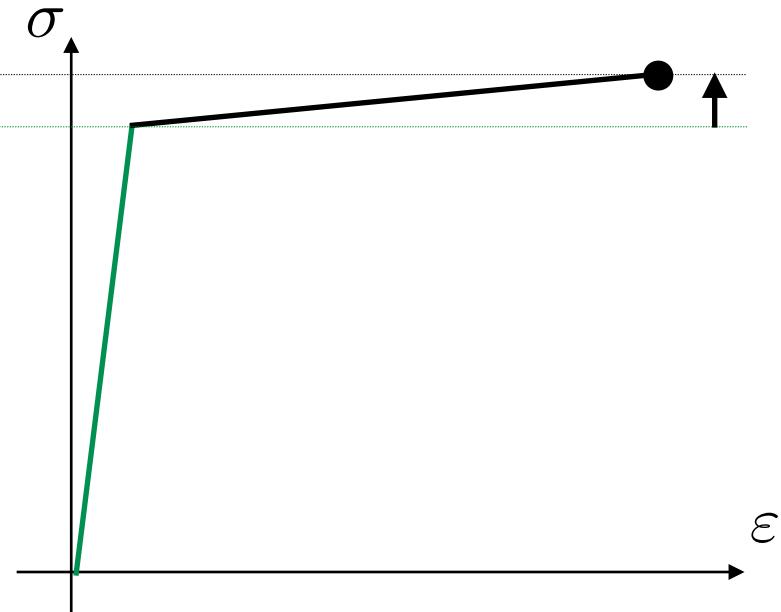
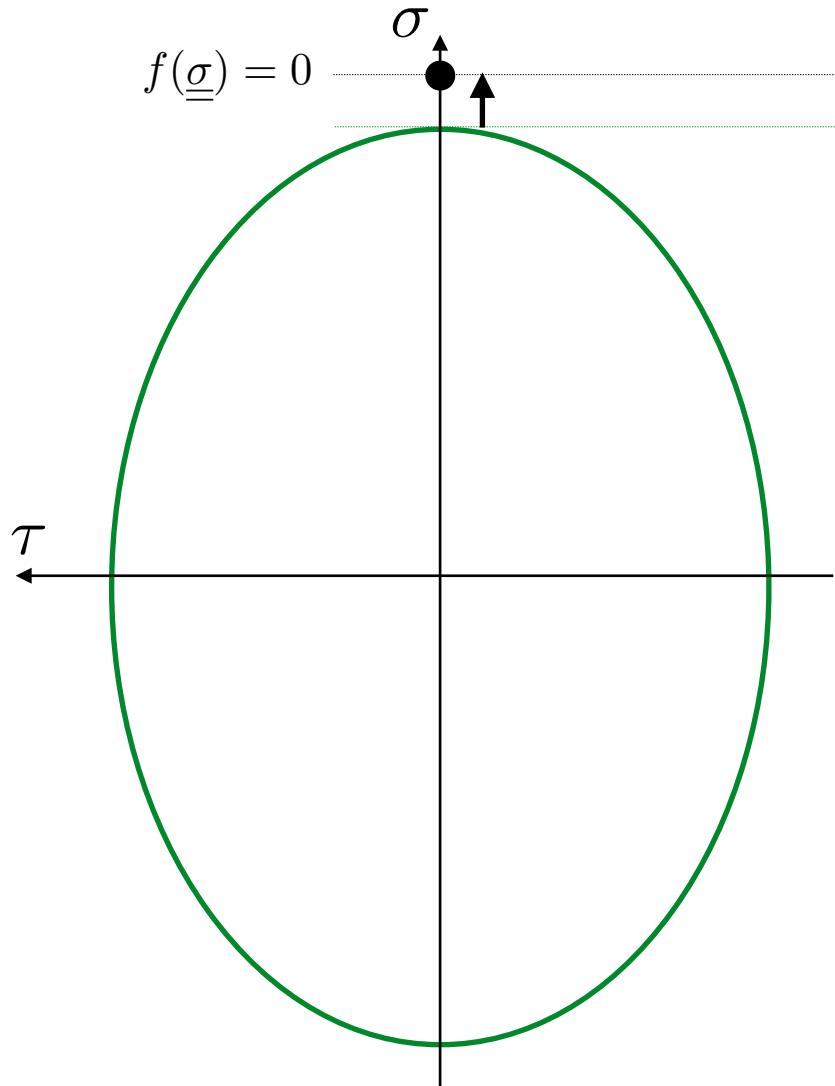
Yield surface evolution in tension-torsion



$$f(\underline{\underline{\sigma}}) = \sqrt{\sigma + 3\tau^2} - \sigma_0$$

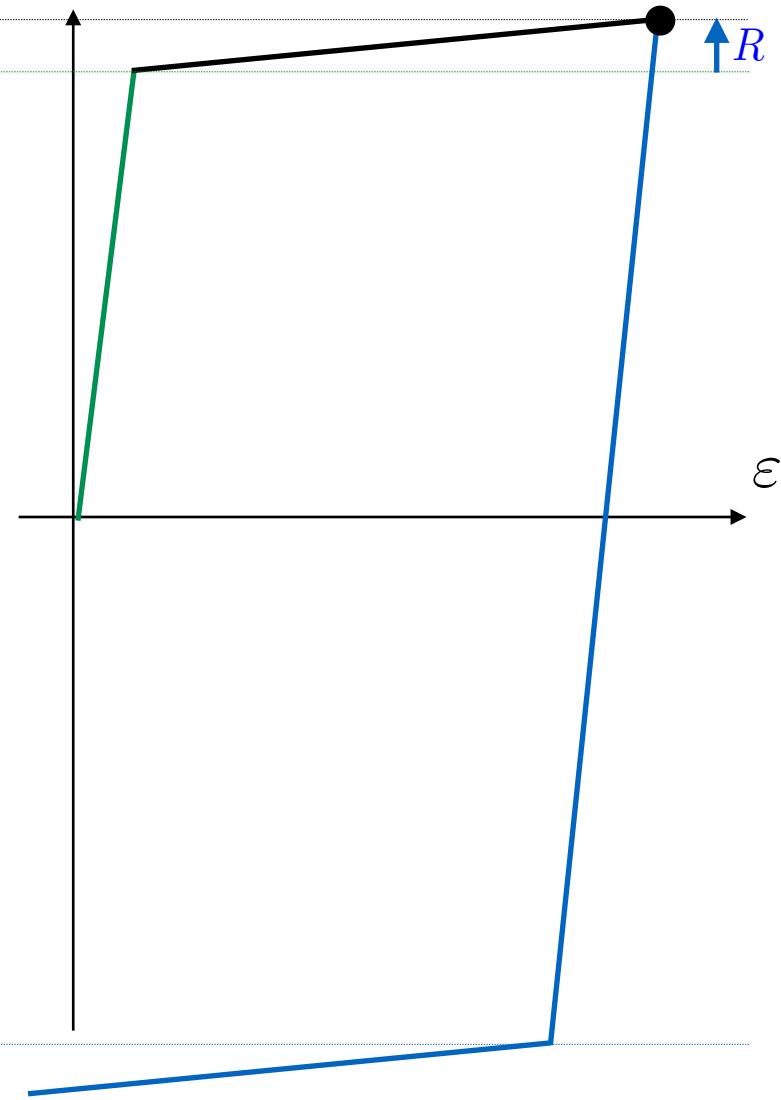
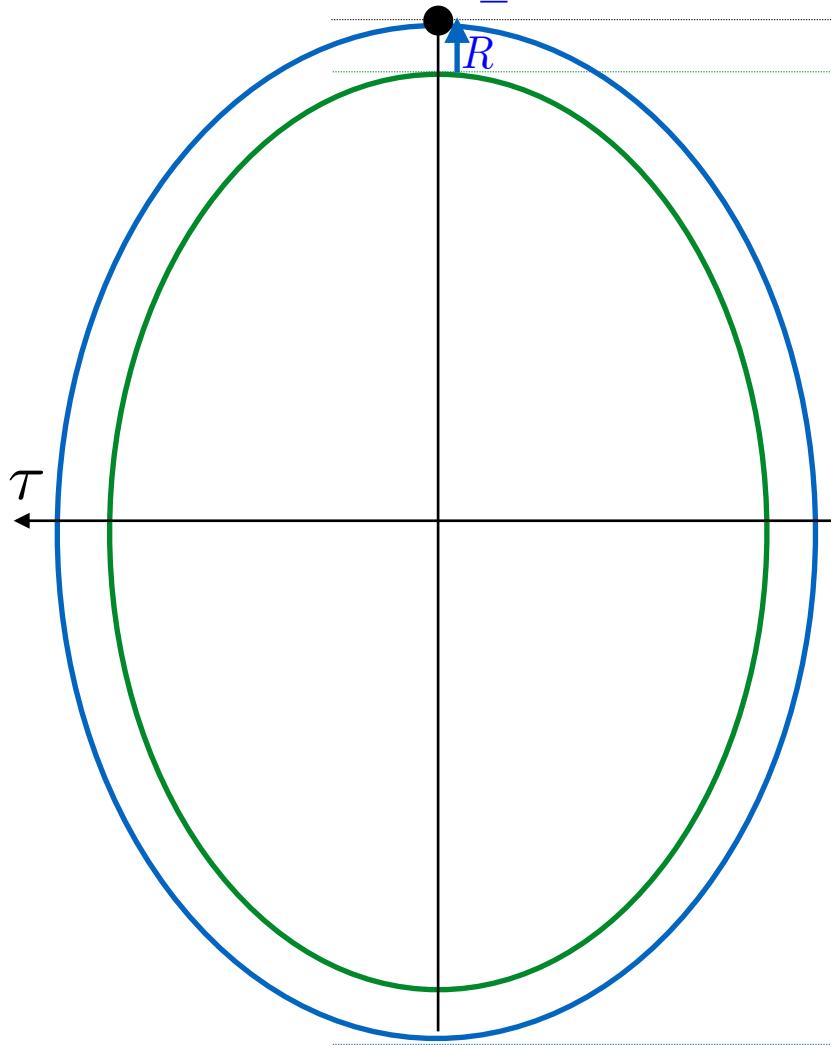


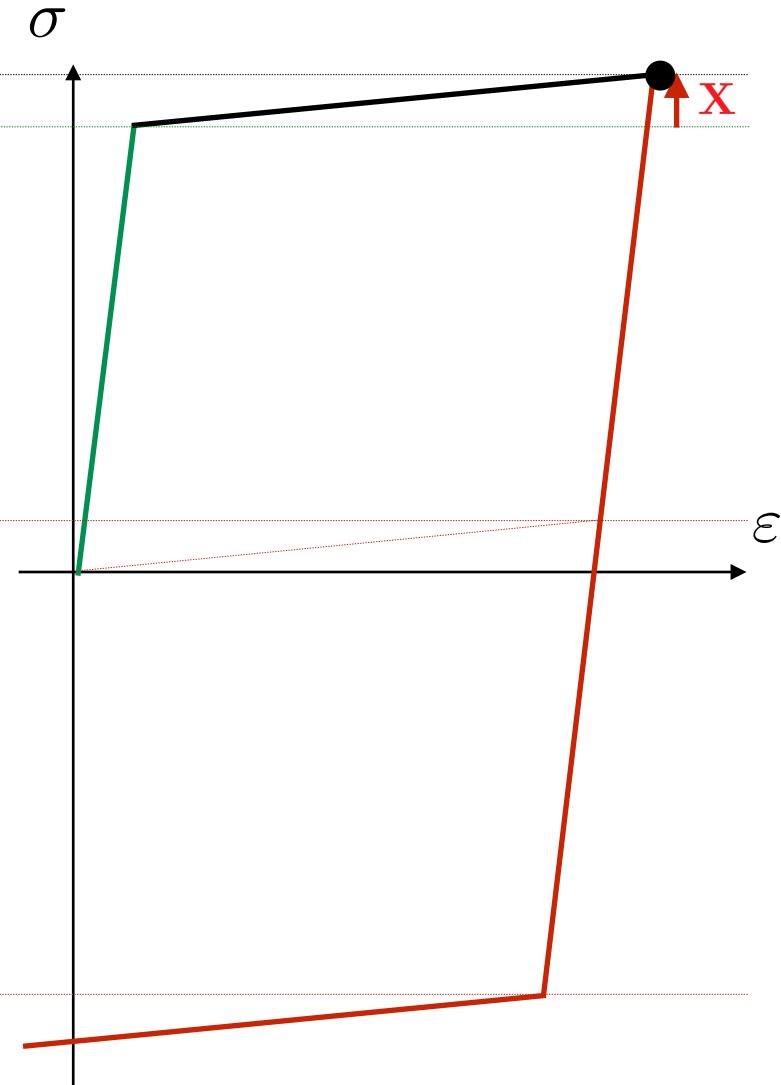
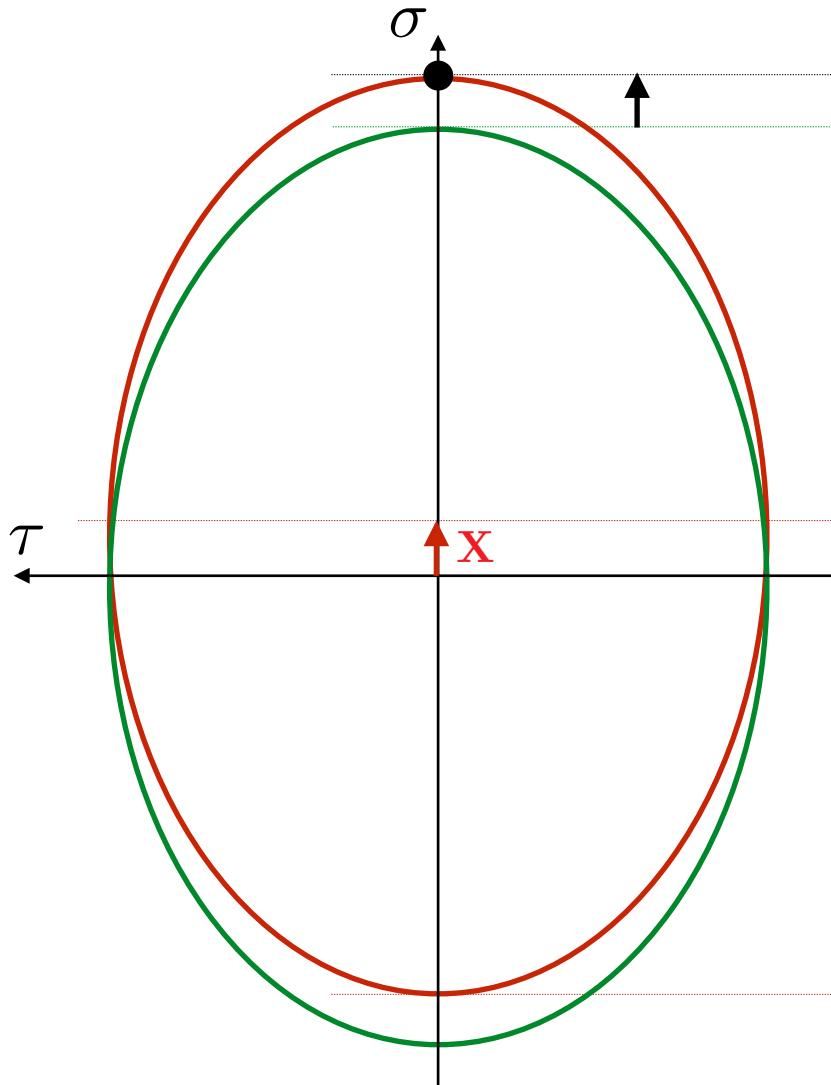




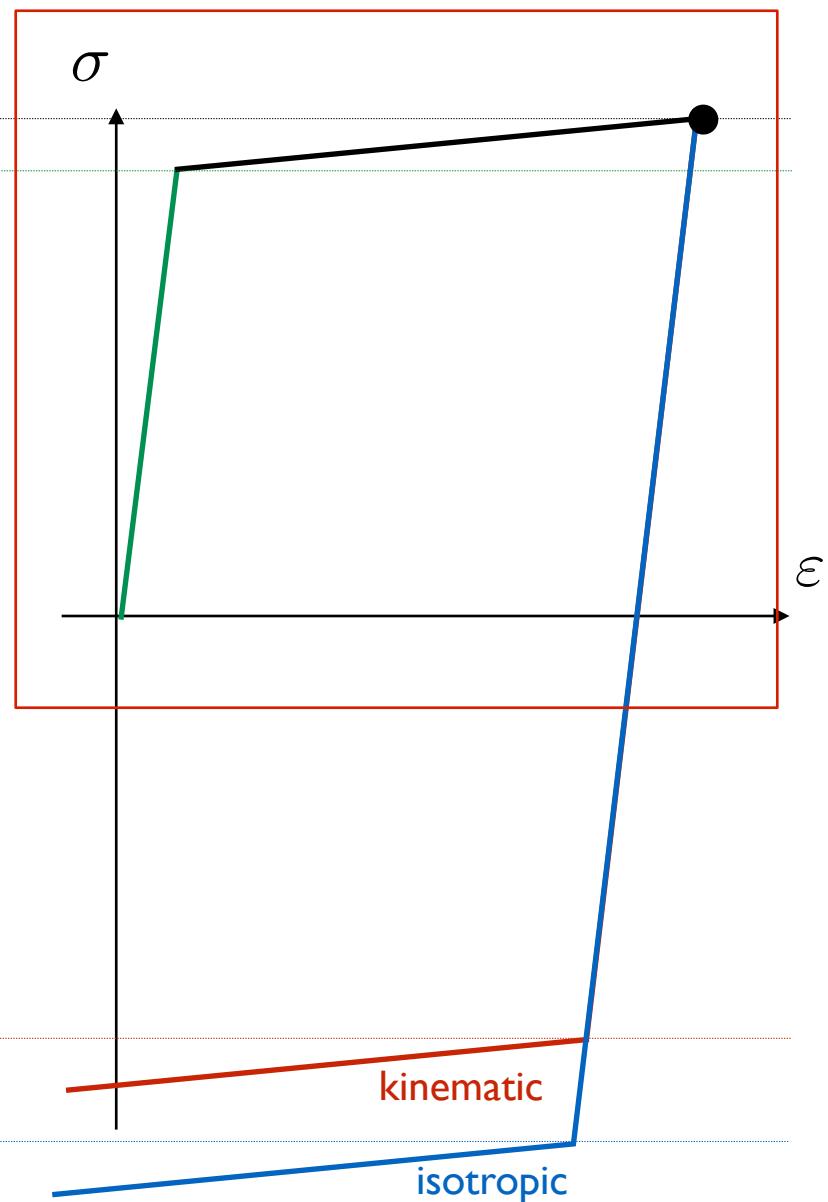
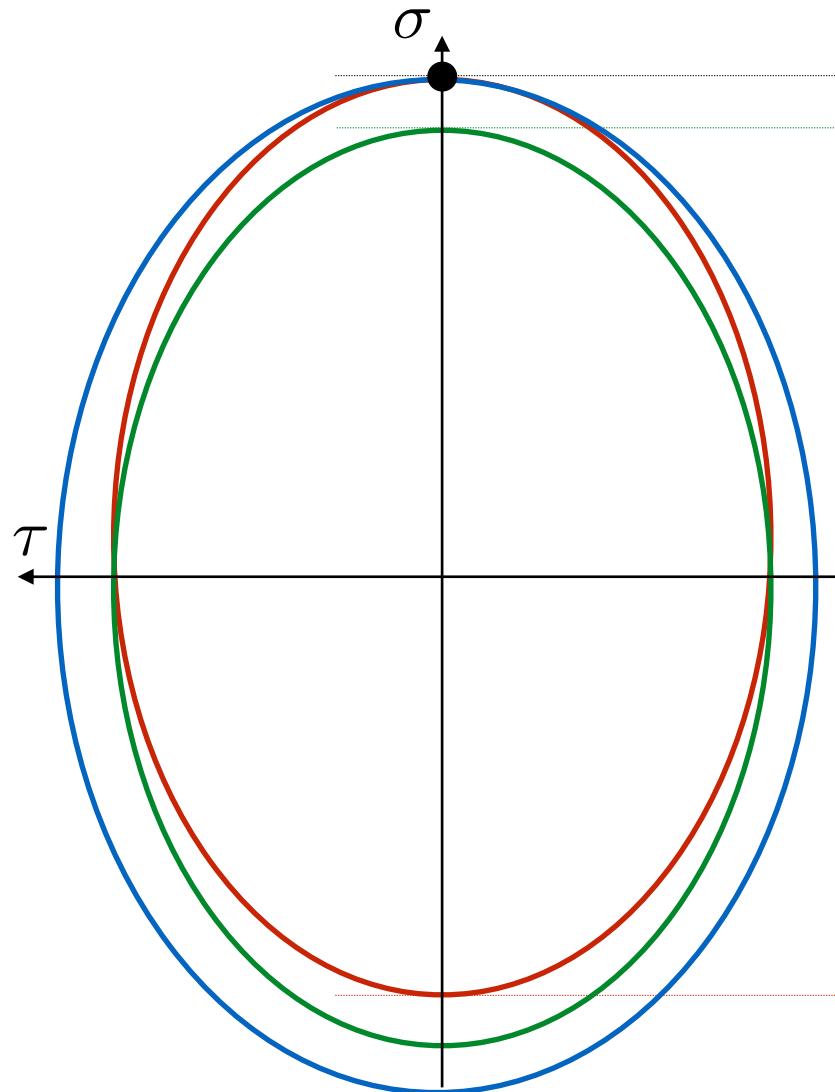
$f(\underline{\sigma}) = 0 \Leftrightarrow$ stay on the yield surface

$$\sigma \ f(\underline{\sigma}) = \sqrt{\sigma + 3\tau^2} - (\sigma_0 + R) \sigma$$





same curves

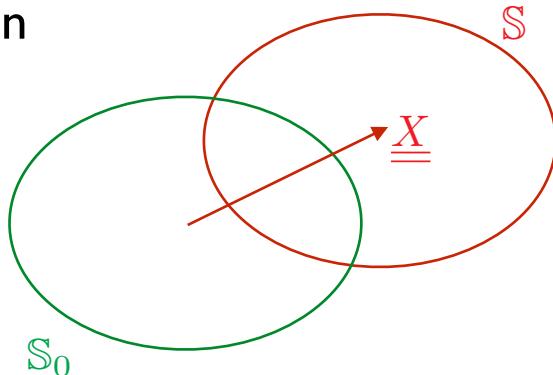


Kinematic hardening

- Definition: translation of the elastic domain

$$\mathbb{S} = \mathbb{S}_0 + \underline{\underline{X}}$$

- One has to precise the evolution of $\underline{\underline{X}}$



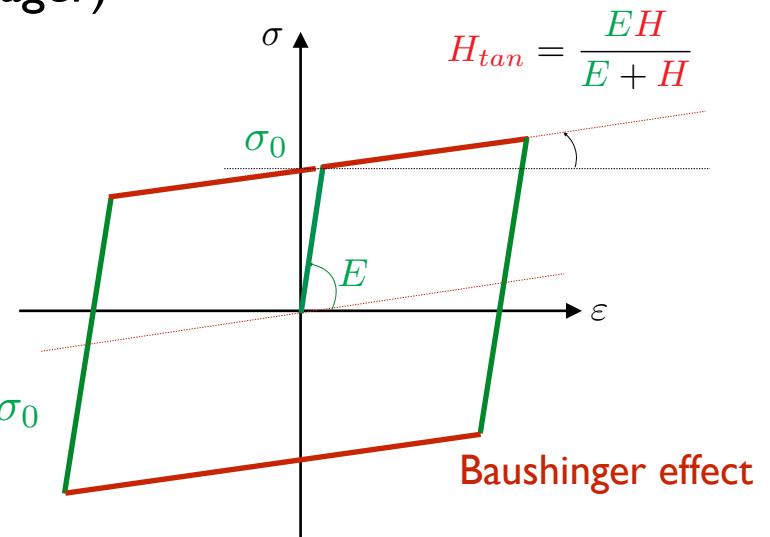
- Example : linear kinematic hardening (Prager)

$$\underline{\underline{X}} = \underline{\underline{H}} : \underline{\underline{\varepsilon}}_p = \frac{2}{3} H \underline{\underline{\varepsilon}}_p$$



W. Prager
(1903-1980)

$$|\sigma - H \varepsilon_p| \leq \sigma_0$$



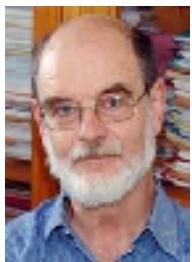
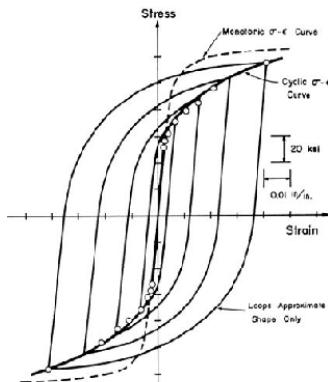
Kinematic hardening

- Non linear kinematic hardening (Armstrong-Frederick, Chaboche)

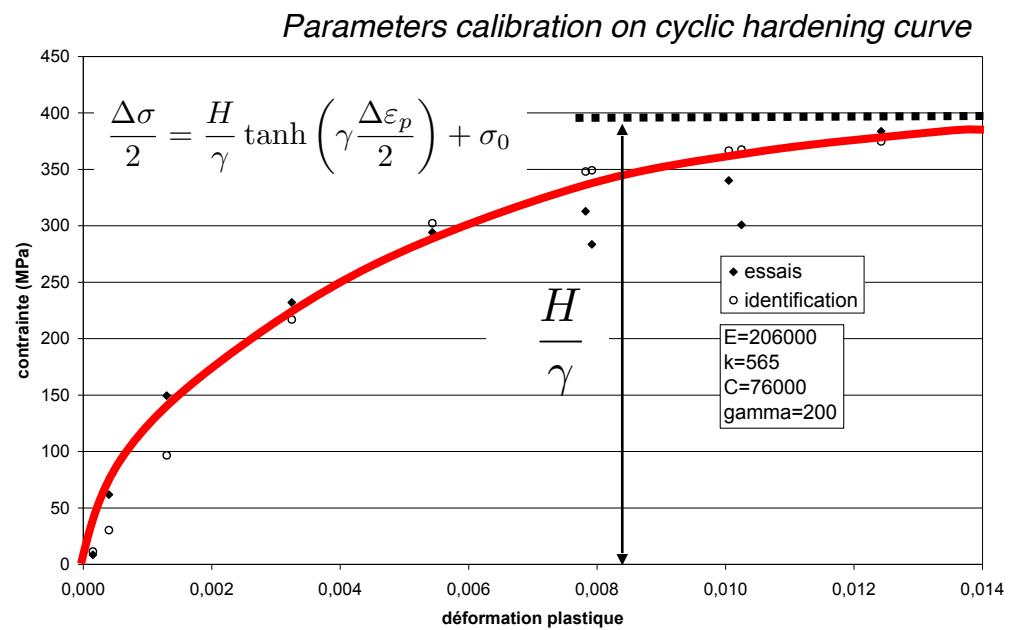
$$f(\underline{\sigma}) = \sqrt{\frac{3}{2}(\underline{s} - \underline{X}) : (\underline{s} - \underline{X})} - \sigma_0$$

$$\dot{\underline{X}} = \frac{2}{3} H \dot{\underline{\varepsilon}_p} - \frac{2}{3} \gamma \underline{X} \dot{p}$$

$$\dot{p} = \sqrt{\frac{2}{3} \dot{\underline{\varepsilon}_p} : \dot{\underline{\varepsilon}_p}}$$



J.-L. Chaboche
(1945-)

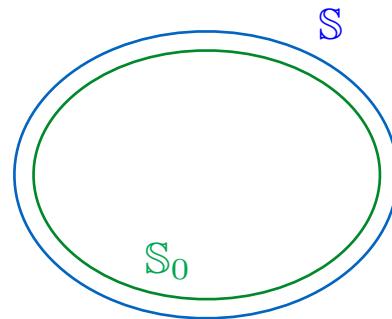


Isotropic hardening

- Definition : dilatation of the elastic domain

$$\mathbb{S} = \mathbb{R} \cdot \mathbb{S}_0$$

- One has to precise the evolution of the yield stress: R



- Example : von Mises with isotropic hardening

$$f(\underline{\sigma}) = \sqrt{\frac{3}{2}\underline{\sigma} : \underline{\sigma} - \frac{1}{2}tr(\underline{\sigma})^2} - \sigma_0 - R(p)$$

p cumulated plastic strain

$$\dot{p} = \sqrt{\frac{2}{3}\dot{\underline{\varepsilon}}_p : \dot{\underline{\varepsilon}}_p}$$

$$p = \int_0^t \dot{p} dt$$

Isotropic hardening

- Example : von Mises with isotropic hardening

$$f(\underline{\underline{\sigma}}) = \sqrt{\frac{3}{2}\underline{\underline{\sigma}} : \underline{\underline{\sigma}} - \frac{1}{2}tr(\underline{\underline{\sigma}})^2} - \sigma_0 - R(p) \quad \dot{p} = \sqrt{\frac{2}{3}\underline{\dot{\underline{\varepsilon}}}_p : \underline{\dot{\underline{\varepsilon}}}_p}$$

- Linear isotropic hardening: $R(p) = H \cdot p$

- Non-linear isotropic hardening:

$$R(p) = R_\infty(1 - \exp(-bp))$$



J. Lemaître
(1934-)

Phenomenological approach

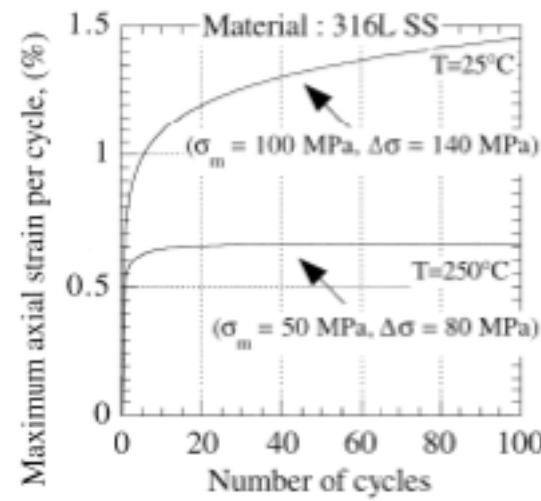
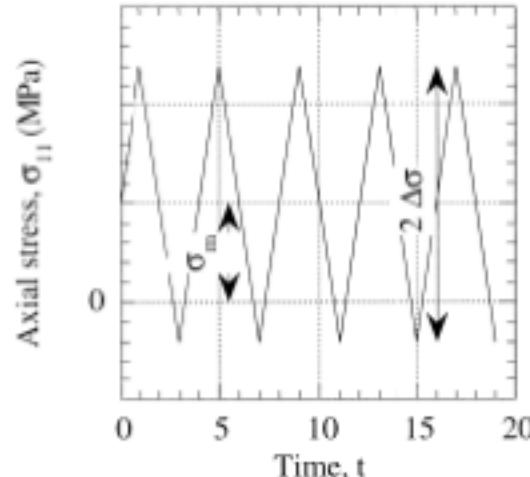
■ Combined hardening:

- non linear isotropic hardening: $R(p) = R_\infty(1 - \exp(-bp))$

- non linear kinematic hardening: $\underline{\dot{X}} = \frac{2}{3}H\underline{\dot{\varepsilon}_p} - \frac{2}{3}\gamma\underline{\underline{X}}\dot{p}$
- Back-stress, hardening saturation, recovering, ...

Combining:
« classical »
approach

- Example: ratcheting modeling



Combined hardening laws

Non-linear kinematic hardenings (Chaboche)

Constitutive equations of the NLK model

Strain decomposition: $\underline{\varepsilon} = \underline{\varepsilon}^e + \underline{\varepsilon}^p$

Hooke's law: $\underline{\varepsilon}^e = \frac{1}{2\mu} \left(\underline{\mathbb{1}} - \frac{1-\nu}{1+\nu} \underline{\mathbb{1}} \otimes \underline{\mathbb{1}} \right) : \underline{\sigma}$ with $\mu = \frac{E}{2(1+\nu)}$

Yield function: $f(\underline{\sigma}, R, \underline{X}) = J_2(\underline{\sigma} - \underline{X}) - R - k$

where $J_2 = \sqrt{\frac{1}{2} (\underline{S} - \underline{X}) : (\underline{S} - \underline{X})}$ and $\underline{S} = \underline{\sigma} - \frac{1}{3} \nu \underline{\sigma} \underline{\mathbb{1}}$

Flow rule: $\underline{\dot{\varepsilon}}^p = \lambda \frac{\underline{\dot{x}}}{\dot{\sigma}_0} = \lambda \underline{\dot{\sigma}}$ with $\underline{\dot{\sigma}} = \frac{1}{2} \frac{\underline{X} - \underline{X}_i}{J_2(\underline{S} - \underline{X})}$

Kinematic hardening rule: $\underline{X} = \underline{X}_1 + \underline{X}_2 - \underline{X}_i - \frac{2}{3} C_i \underline{\dot{\varepsilon}}^p - \gamma_i \varphi(p) \rho \underline{X}_i \quad (i = 1, 2)$

with $\varphi(p) = \varphi_\infty + (1 - \varphi_\infty) e^{-ap}$

Isotropic hardening rule: $\dot{R} = b(Q_\infty - R)\rho$

with $k, Q_\infty, b, \varphi_\infty, a, C_1, C_2, \gamma_1, \gamma_2$ material parameters

Non-linear kinematic hardenings (Ohno-Wang)

Constitutive equations of the OW model

Strain decomposition: $\underline{\varepsilon} = \underline{\varepsilon}^e + \underline{\varepsilon}^p$

Hooke's law: $\underline{\varepsilon}^e = \frac{1}{2\mu} \left(\underline{\mathbb{1}} - \frac{1-\nu}{1+\nu} \underline{\mathbb{1}} \otimes \underline{\mathbb{1}} \right) : \underline{\sigma}$ with $\mu = \frac{E}{2(1+\nu)}$

Yield function: $f(\underline{\sigma}, R, \underline{X}) = J_2(\underline{\sigma} - \underline{X}) - R - k$

where $J_2 = \sqrt{\frac{1}{2} (\underline{S} - \underline{X}) : (\underline{S} - \underline{X})}$ and $\underline{S} = \underline{\sigma} - \frac{1}{3} \nu \underline{\sigma} \underline{\mathbb{1}}$

Flow rule: $\underline{\dot{\varepsilon}}^p = \lambda \frac{\underline{\dot{x}}}{\dot{\sigma}_0} = \lambda \underline{\dot{\sigma}}$ with $\underline{\dot{\sigma}} = \frac{1}{2} \frac{\underline{X} - \underline{X}_i}{J_2(\underline{S} - \underline{X})}$

and $I_i = \frac{C_i}{\gamma_i \varphi(p)} \bar{X}_i = \sqrt{\frac{3}{2} (\bar{X}_i : \underline{X}_i)} \quad \bar{k}_i = \frac{\underline{X}_i}{\bar{X}_i}$

Kinematic hardening rule: $\underline{X} = \underline{X}_1 + \underline{X}_2 - \underline{X}_i - \frac{2}{3} C_i \underline{\dot{\varepsilon}}^p - \lambda_i \varphi(p) \left(\frac{\bar{X}_i}{\lambda_i} \right)^{m_i} \left(\underline{\dot{\varepsilon}}^p : \bar{k}_i \right) \bar{X}_i \quad (i = 1, 2)$

with $\varphi(p) = \varphi_\infty + (1 - \varphi_\infty) e^{-ap}$

where $\langle \cdot \rangle$ are the Max Cauley brackets: $\langle u \rangle = 0$ if $u > 0$ and $\langle u \rangle = 0$ if $u < 0$

Isotropic hardening rule: $\dot{R} = b(Q_\infty - R)\rho$

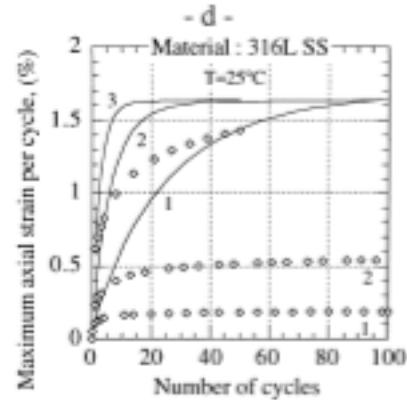
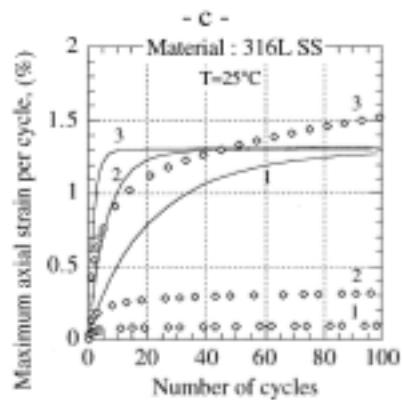
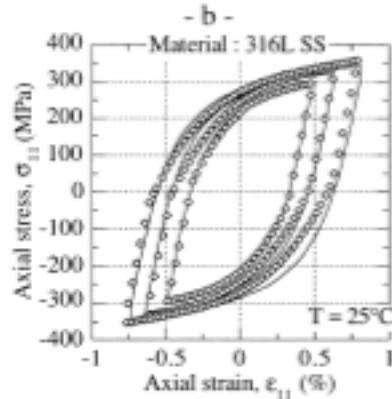
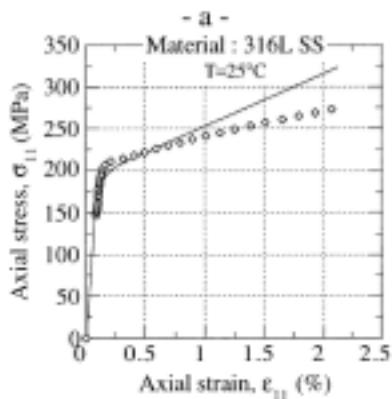
with $k, Q_\infty, b, \varphi_\infty, a, C_1, C_2, \gamma_1, \gamma_2, m_1, m_2$ material parameters

+ Burlet-Cailletaud + Tanaka

[Portier et al., Ratchetting under tension-torsion loadings: experiments and modelling, IJP, 2000]

Influence of hardening rule on ratcheting

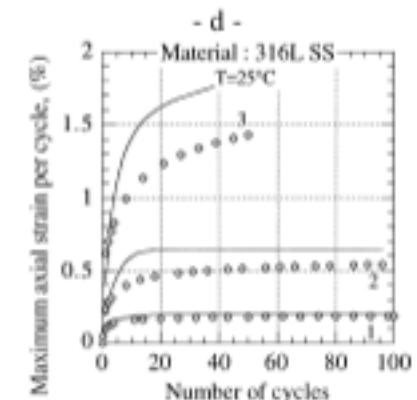
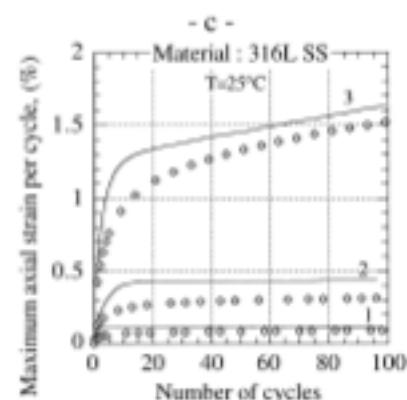
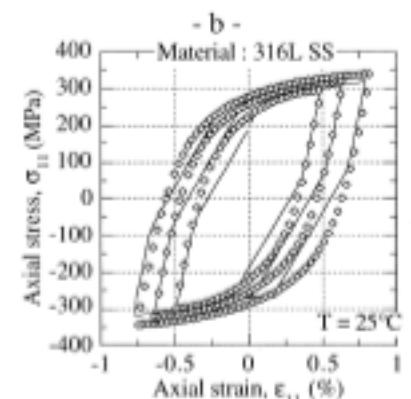
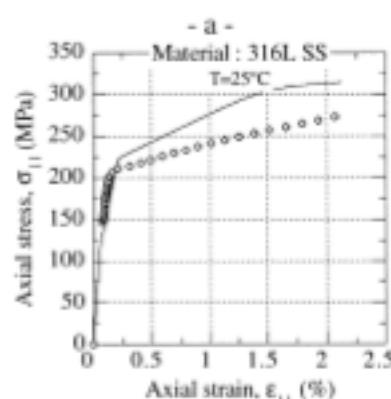
Non-linear kinematic hardenings (Chaboche)



Tension-torsion ratcheting tests

[Portier et al., 2000]

Non-linear kinematic hardenings (Ohno-Wang)



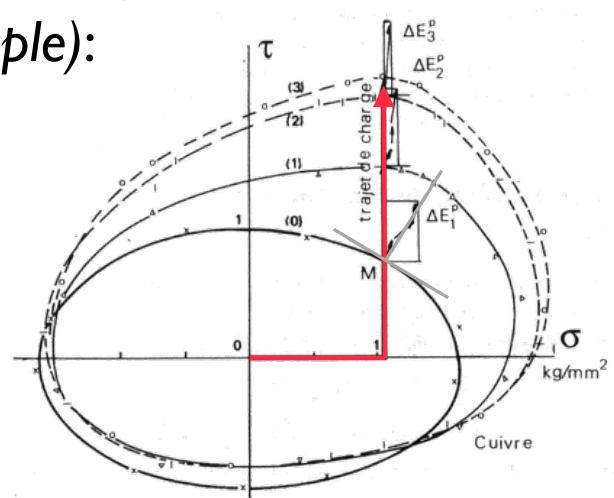
Tension-torsion ratcheting tests

See also works of Calloch, Vincent, ... Yield surface distorsion, ...

Some theoretical justifications

- From experiments (see *Taylor, Bui* for example):

- **convexity** yield surface
- **normality** law



- **Drücker-Ilyushin postulate:**

- The deformation work has to be positive in the case of a *strain cycle* respecting the evolution law.

$$\Delta W = \int_{t_0}^{t_1} \underline{\underline{\sigma}} : \underline{\underline{\dot{\varepsilon}}} dt \geq 0$$

- Strain cycle: $\underline{\underline{\varepsilon}}(t_1) = \underline{\underline{\varepsilon}}(t_0)$

Some theoretical justifications

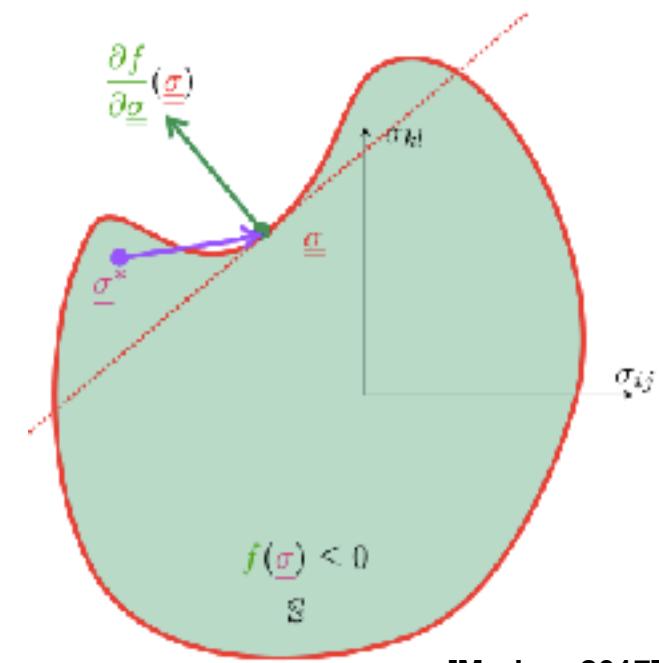
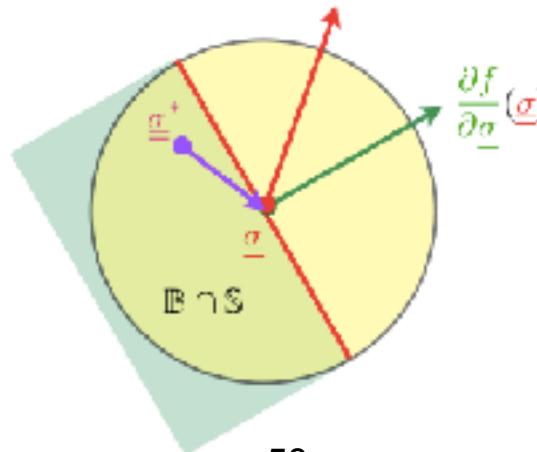
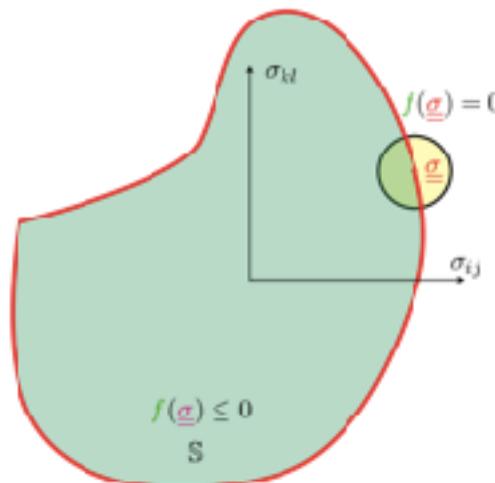
■ Maximal dissipation principle (Hill, demonstration in Marigo for ex.)

- Inequality deduced from Drücker-Ilyushin postulate in perfect plasticity:

$$(\underline{\sigma}(t) - \underline{\sigma}^*) : \dot{\underline{\varepsilon}}^p(t) \geq 0 \quad \forall \underline{\sigma}^* \in \mathbb{S}$$

[Hill, 1950]

- Implies:
 - **convexity** of the yield surface
 - **normality** law



Thermodynamics of Irreversible Processes ?

[Germain, 1973, Germain, Nguyen, Suquet, 1983]

- Standard Generalized Materials (Halphen and Nguyen, 1975)
 - **free energy potential:** state equations

$$w(\underline{\underline{\varepsilon}}, \underline{\underline{\varepsilon}}^p, \underline{\underline{\alpha}}) = w_e(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^p) + w_\alpha(\underline{\underline{\alpha}})$$



P. Germain
(1920-2009)

- **dissipation potential:** complementary equations

$$\varphi(\dot{\underline{\underline{\varepsilon}}}^p, \dot{\underline{\underline{\alpha}}}, \underline{\underline{\varepsilon}}, \underline{\underline{\varepsilon}}^p, \underline{\underline{\alpha}}) \quad \text{and} \quad (\underline{\underline{A}}_{\varepsilon^p}, \underline{\underline{A}}_\alpha) \in \partial_{(\dot{\underline{\underline{\varepsilon}}}^p, \dot{\underline{\underline{\alpha}}})} \varphi(\dot{\underline{\underline{\varepsilon}}}^p, \dot{\underline{\underline{\alpha}}}; \underline{\underline{\varepsilon}}, \underline{\underline{\varepsilon}}^p, \underline{\underline{\alpha}})$$



P. Suquet
(1954-)

- dual dissipation potential: Legendre-Fenchel transform

$$\varphi^*(\underline{\underline{A}}_{\varepsilon^p}, \underline{\underline{A}}_\alpha) = \sup_{\dot{\underline{\underline{\varepsilon}}}^p, \dot{\underline{\underline{\alpha}}}} \left\{ \underline{\underline{A}}_{\varepsilon^p} : \dot{\underline{\underline{\varepsilon}}}^p + \underline{\underline{A}}_\alpha : \dot{\underline{\underline{\alpha}}} - \varphi(\dot{\underline{\underline{\varepsilon}}}^p, \dot{\underline{\underline{\alpha}}}) \right\}$$



B. Halphen

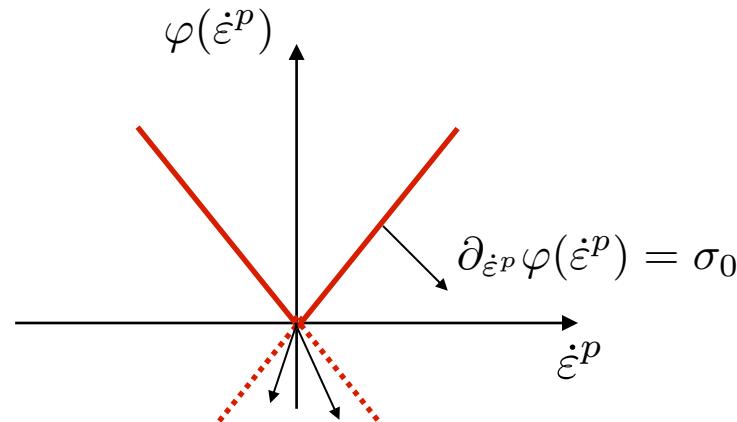
$$(\dot{\underline{\underline{\varepsilon}}}^p, \dot{\underline{\underline{\alpha}}}) \in \partial_{(\underline{\underline{A}}_{\varepsilon^p}, \underline{\underline{A}}_\alpha)} \varphi^*(\underline{\underline{A}}_{\varepsilon^p}, \underline{\underline{A}}_\alpha)$$



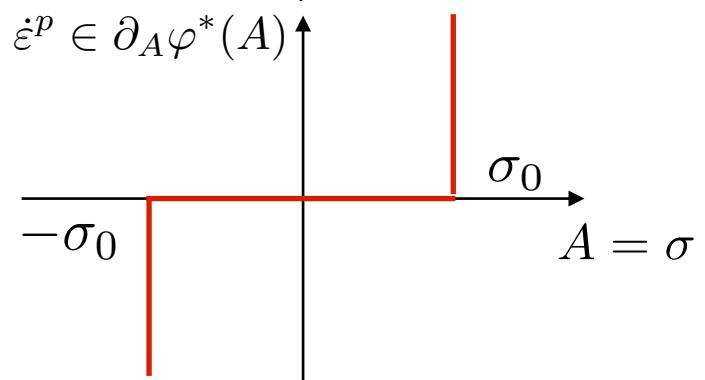
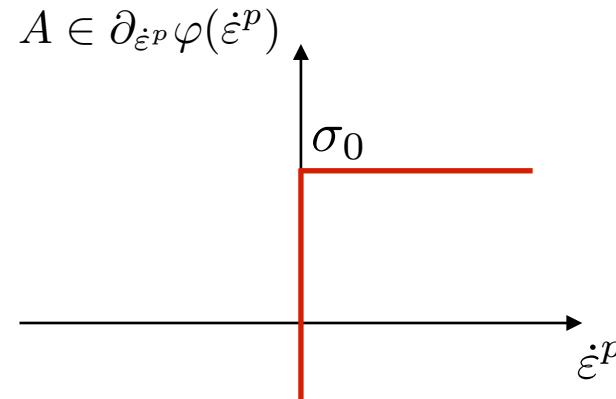
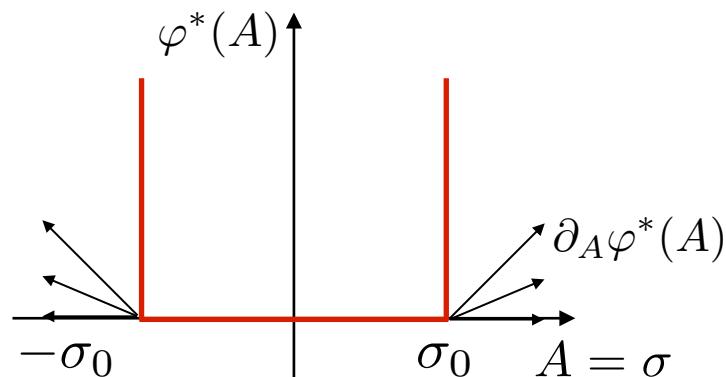
Q.S. Nguyen

Potential, dual potential and sub-differential

- Example: uniaxial perfect plasticity $\varphi(\dot{\varepsilon}^p) = \sigma_0 |\dot{\varepsilon}^p|$



$$\varphi^*(A) = \sup_{\dot{\varepsilon}^p} \{ \sigma \dot{\varepsilon}^p - \sigma_0 |\dot{\varepsilon}^p| \} = \sup_{|\dot{\varepsilon}^p|} \{ (\sigma - \sigma_0) |\dot{\varepsilon}^p| \}$$



Some equivalences

- Dual potential:

- indicator of a convex set:

$$|\sigma| - \sigma_0 \leq 0$$

- sub-gradient:

$\dot{\varepsilon}^p$ du signe de σ_0

- Equivalence with:

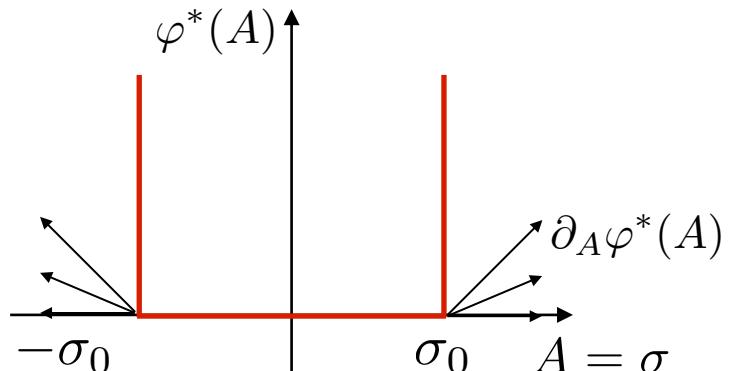
- convex yield surface:

$$f(\sigma) = |\sigma| - \sigma_0$$

- normality law

$$\dot{\varepsilon}^p = \dot{\lambda} \operatorname{signe}(\sigma_0) \quad \dot{\lambda} \geq 0$$

$$\varphi^*(A) = \sup_{\dot{\varepsilon}^p} \{ \sigma \dot{\varepsilon}^p - \sigma_0 |\dot{\varepsilon}^p| \} = \sup_{|\dot{\varepsilon}^p|} \{ (|\sigma| - \sigma_0) |\dot{\varepsilon}^p| \}$$



**Then, what is the interest of TPI? ;-)
Positive dissipation?**

(Only?) interest of thermodynamics: the energy balance and the heat coupled equation

■ The energy balance and the heat coupled equation

■ The energy balance:

$$\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}} = \rho(\dot{e} - T\dot{s}) + D_1 = \rho(\dot{w} + s\dot{T}) + D_1$$

Provided « work » = Recoverable « work » + Intrinsic dissipation

■ The heat coupled equation:

$$\rho C_\varepsilon \dot{T} + \operatorname{div}(\underline{\underline{q}}) = \underline{r} + \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p - \underline{\underline{A}}_\alpha : \dot{\underline{\underline{\alpha}}} + \left(T \frac{\partial \underline{\underline{\sigma}}}{\partial T} : \dot{\underline{\underline{\varepsilon}}}^e + T \frac{\partial \underline{\underline{A}}_\alpha}{\partial T} : \dot{\underline{\underline{\alpha}}} \right)$$

- external heat supply (volume / surface)
- intrinsic dissipation
- reversible thermomechanical couplings

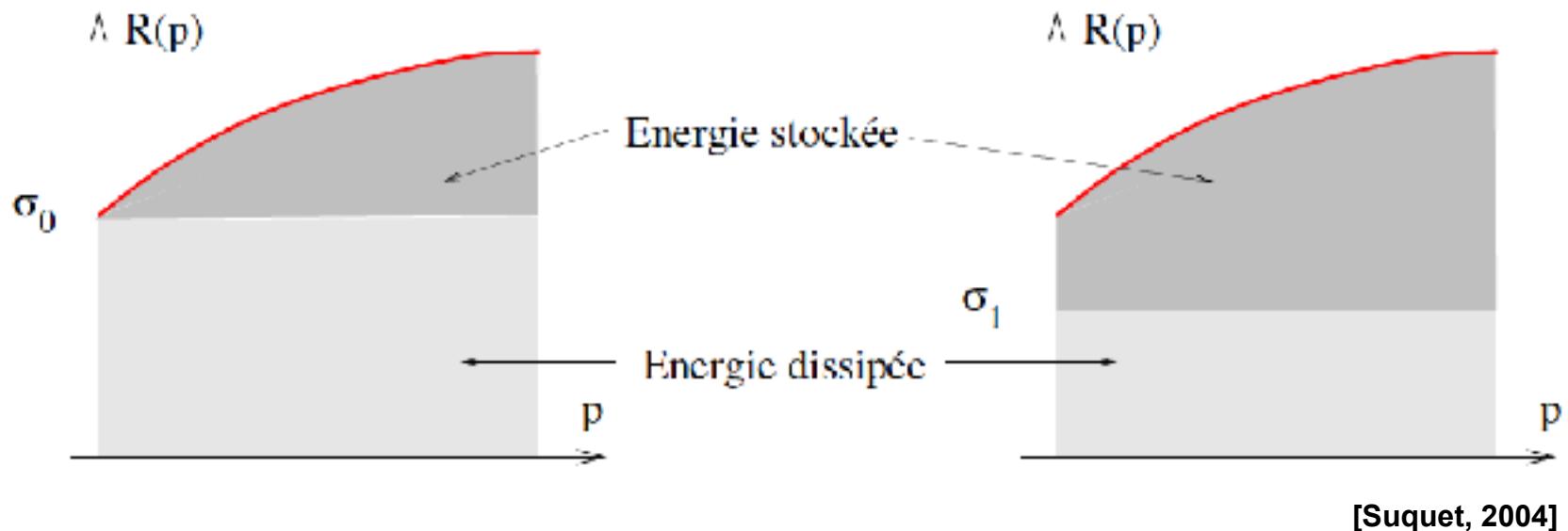
« Classical » thermoelastoplasticity

- The heat coupled equation:

$$\rho \mathcal{C}_\varepsilon \dot{T} + \text{div}(\underline{q}) = \underbrace{\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p - \underline{\underline{A}}_\alpha : \dot{\underline{\underline{\alpha}}}}_{D_1} - T \frac{E\alpha}{1-2\nu} \text{tr}(\dot{\underline{\underline{\varepsilon}}}^e)$$

- no external heat supply in volume
- no coupling between hardening and temperature (no phase transformation for ex.)
- a strong restriction for models:
 - kinematic hardening: $D_1 = \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p - \underline{\underline{X}} : \dot{\underline{\underline{\alpha}}}$
 - isotropic hardening: $D_1 = \underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p - R \dot{p}$

Mechanical equivalence / Energetic equivalence



- **Towards temperature measurements** for thermomechanical sources determination (« à la Chrysochoos », see later)

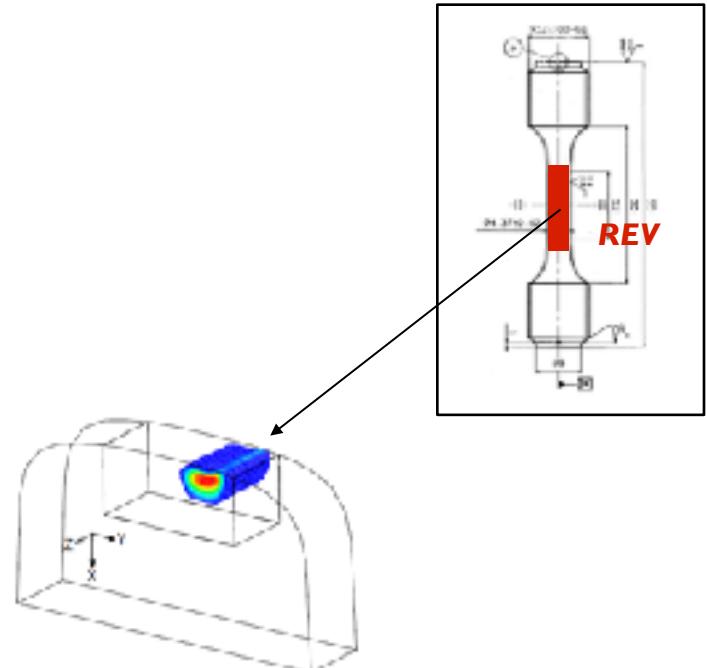
Plasticity: structure scale

■ Material scale:

- question: when does plasticity occur?
 - from uniaxial to multiaxial plasticity: plastic criterion versus yield stress
- question: how does plastic flow occur?
 - flow rule: hardening concept

■ Structure scale:

- structural hardening
 - impact of stress/strain heterogeneities
- residual stresses
 - impact of strain incompatibilities



What is structural hardening?

■ Example of a three bars truss

- Bars only in tension-compression, section A
- Equilibrium:

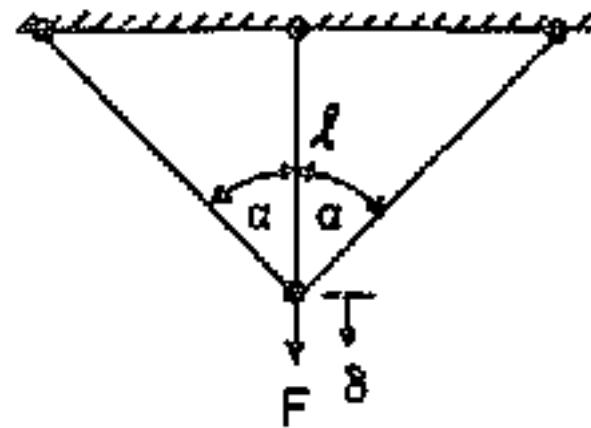
$$\sum_{i=1}^3 N_i t_i = \underline{F}$$

- Yield criterion: $f(\underline{\sigma}) = |\sigma| - \sigma_0$

$$|N_i| \leq N_0$$

■ Elastic solution:

- $F = \frac{AE\delta}{l}(1 + 2 \cos^3 \alpha)$ for $\delta \leq \frac{N_0 l}{AE}$



[Leckie et al., 1990]

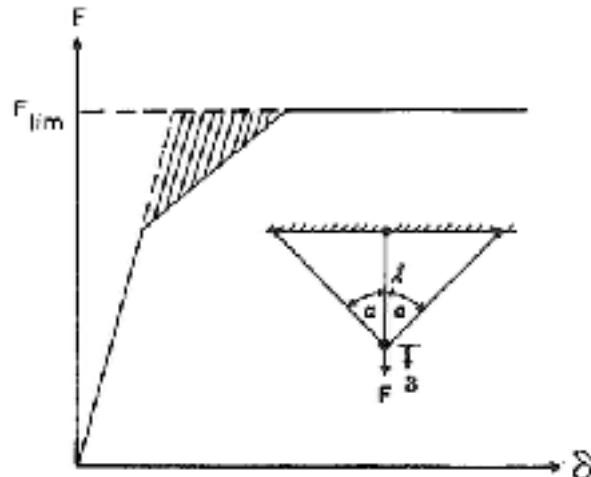
Plasticity of the three-bars truss

- Plasticity of central bar:

- $F = N_0 + \frac{2AE\delta}{l} \cos^3 \alpha$ for

$$\frac{N_0 l}{A E} \leq \delta \leq \frac{N_0 l}{A E \cos^2 \alpha}$$

- Limit load:



[Leckie et al., 1990]

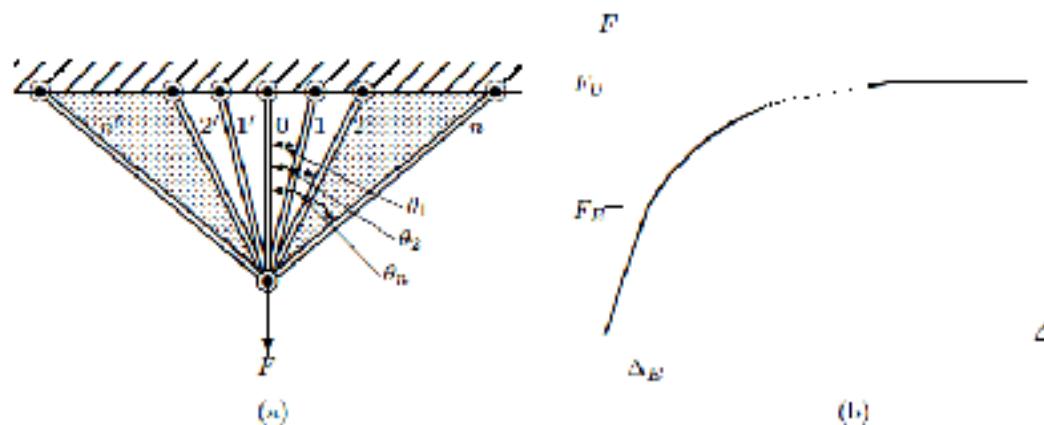
- $F = F_{lim} = N_0(1 + 2 \cos \alpha)$ for

$$\delta \geq \frac{N_0 l}{A E \cos^2 \alpha}$$

- Despite local perfect plasticity, **structural hardening** due to strain incompatibilities and residual stresses

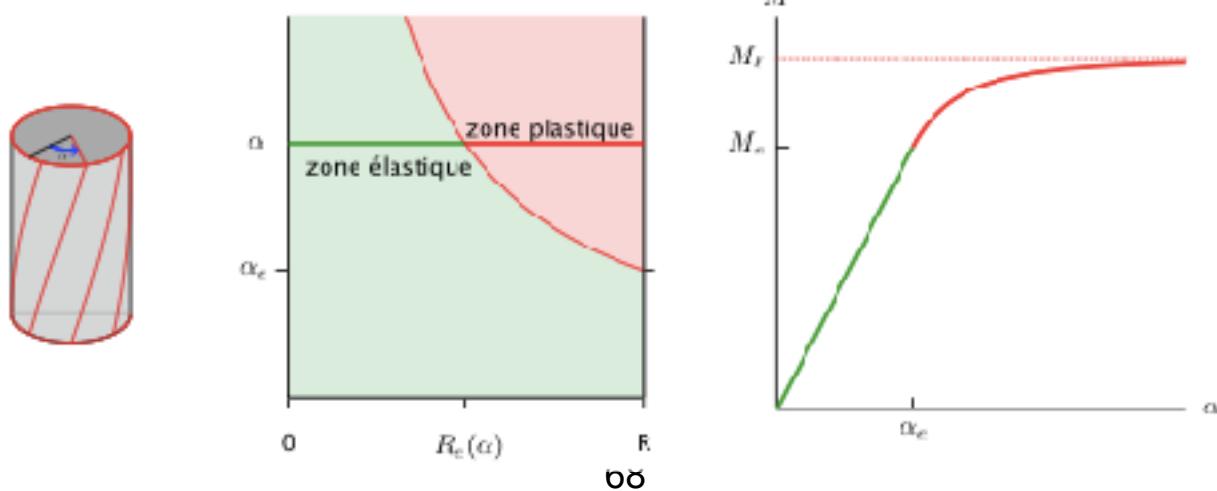
Generalization

- Infinite number of bars:



[Lubliner, 2008]

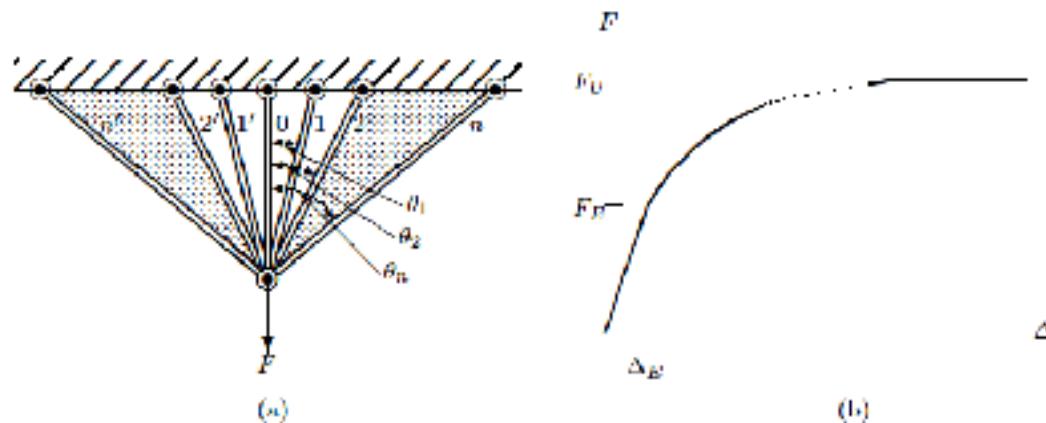
- Tube in torsion:



[Marigo, 2017]

Generalization

- Infinite number of bars:



- Main conclusions:

- Structural hardening = **kinematic hardening**
- Forces in bars: **localization** of the global forces and **residual stresses**

$$\underline{\underline{\sigma}} = \underline{\underline{A}} : \underline{\underline{\Sigma}} + \underline{\underline{\sigma}}^{res}$$

- See *homogenization of polycrystals* for example

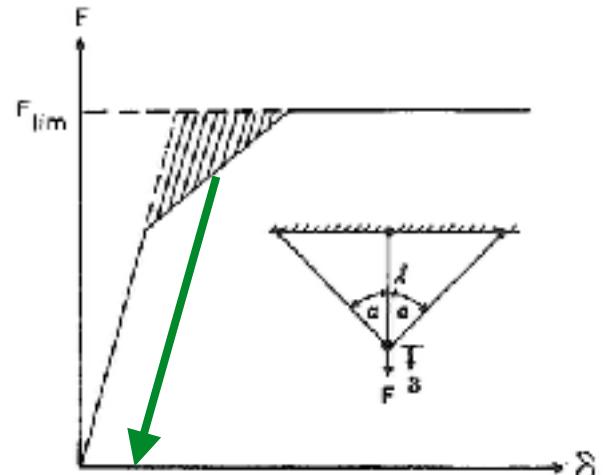
Residual stresses

■ Residual stresses in bars:

- $\sigma_v = -2 \frac{N_0}{A} \frac{\sin^2 \alpha \cos \alpha}{1 + 2 \cos^3 \alpha}$ for the vertical bar (compression),
- $\sigma_i = \frac{N_0}{A} \frac{\sin^2 \alpha}{1 + 2 \cos^3 \alpha}$ for the inclined bar (tension)

■ Generalization:

- **self-equilibrated** stress field:
 $\operatorname{div} \underline{\underline{\sigma}}^{res} = 0$ sur Ω and $\underline{\underline{\sigma}}^{res} \cdot \underline{n} = 0$ on S_T
- one can (easily) prove that: $\int_{\Omega} \operatorname{tr}(\underline{\underline{\sigma}}) d\Omega = 0$
 - some zones **in tension** + some zones **in compression**



Real and elastic solutions

- The **real solution** is composed with:
 - a stress field $\underline{\underline{\sigma}}$ statically admissible and plastically admissible
(i.e. $f(\underline{\underline{\sigma}}) \leq 0$)
 - a strain field $\underline{\underline{\varepsilon}}$ kinematically admissible and a plastic strain field $\underline{\underline{\varepsilon}}^p$
 - as: $\underline{\underline{\sigma}} = \underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^p)$
- The **pure elastic solution** is composed with:
 - a stress field $\underline{\underline{\sigma}}^{el}$ statically admissible
 - a strain field $\underline{\underline{\varepsilon}}^{el}$ kinematically admissible
 - as: $\underline{\underline{\sigma}}^{el} = \underline{\underline{C}} : \underline{\underline{\varepsilon}}^{el}$

Residual solution

- The **residual solution is defined by the difference between the real solution and the elastic one:**

$$\left\{ \begin{array}{ccc} \text{SA0} & \text{SA} & \text{SA} \\ \underline{\underline{\sigma}}^{res}(M, t) = \underline{\underline{\sigma}}(M, t) - \underline{\underline{\sigma}}^{el}(M, t) \\ \underline{\underline{\varepsilon}}^{res}(M, t) = \underline{\underline{\varepsilon}}(M, t) - \underline{\underline{\varepsilon}}^{el}(M, t) \\ \text{KA0} & \text{KA} & \text{KA} \end{array} \right.$$

- Therefore:

- $\underline{\underline{\sigma}}^{res}$ is **SA0** and $\underline{\underline{\varepsilon}}^{res}$ is **KA0**

- $\underline{\underline{\sigma}}^{res} = \underline{\underline{\underline{C}}} : (\underline{\underline{\varepsilon}}^{res} - \underline{\underline{\varepsilon}}^p)$ as

$$\begin{aligned}\underline{\underline{\sigma}}^{res} &= \underline{\underline{\sigma}} - \underline{\underline{\sigma}}^{el} \\ \underline{\underline{\sigma}}^{res} &= \underline{\underline{\underline{C}}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^p) - \underline{\underline{\underline{C}}} : \underline{\underline{\varepsilon}}^{el} \\ \underline{\underline{\sigma}}^{res} &= \underline{\underline{\underline{C}}} : (\underline{\underline{\varepsilon}}^{res} - \underline{\underline{\varepsilon}}^p)\end{aligned}$$

Residual stresses

- Therefore:

- $\operatorname{div} \underline{\underline{\sigma}}^{res} = 0$ on Ω
- $\underline{\underline{\sigma}}^{res} \cdot \underline{n} = 0$ on S_T SA0
- $\underline{\underline{\sigma}}^{res} = \underline{\underline{C}} : (\underline{\underline{\varepsilon}}^{res} - \underline{\underline{\varepsilon}}^p)$
- $\underline{\underline{\varepsilon}}^{res} = \frac{1}{2}(\underline{\underline{\operatorname{grad}}} \underline{u}^{res} + {}^T \underline{\underline{\operatorname{grad}}} \underline{u}^{res})$ KA0
- $\underline{u}^{res} = 0$ on S_u

- The residual stress field is solution of an elastic problem with an initial strain field $\underline{\underline{\varepsilon}}^p$ and zero forces.
- **To compute residual stress field:**

$$\underline{\underline{\sigma}}^{res}(M, t) = \underline{\underline{\sigma}}(M, t) - \underline{\underline{\sigma}}^{el}(M, t)$$

Elastic unloading

Incompatibilities

- If $\underline{\underline{\varepsilon}}^p$ is compatible:
 - then it exists \underline{u}^p as: $\underline{\underline{\varepsilon}}^p = \frac{1}{2}(\underline{\text{grad}}\underline{u}^p + {}^T\underline{\text{grad}}\underline{u}^p)$
 $\underline{u}^p = 0$ on S_u
 - then $\underline{u}^{res} = \underline{u}^p$ (uniqueness of the elastic solution) and $\underline{\underline{\sigma}}^{res} = 0$
- If $\underline{\underline{\varepsilon}}^p$ is not compatible:
 - then, there exists $\underline{\underline{\varepsilon}}_{res}^{el}$ as: $\underline{\underline{\varepsilon}}^{res} = \underline{\underline{\varepsilon}}_{res}^{el} + \underline{\underline{\varepsilon}}^p$
 - in order to obtain $\underline{\underline{\varepsilon}}^{res}$ KA0

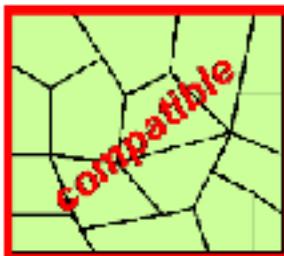
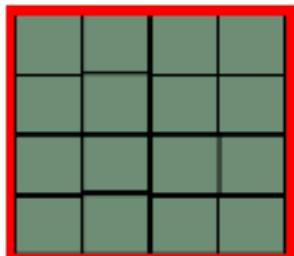


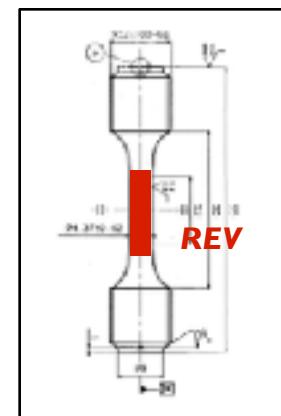
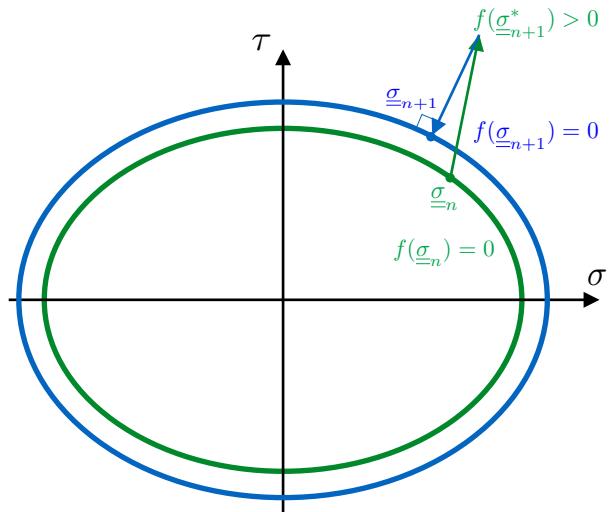
Illustration: watermelon and rhubarb



Two scales

- Material scale
 - **yield surface**
 - **strain hardening**
 - Structure scale
 - **structural hardening**
 - **residual stresses**
 - Numerical aspects
 - Material scale : **radial-return**
 - Structure scale : **direct algorithms**
 - Some illustrations, synthesis, remarks and references
- Constitutive models?
- Parameters calibration?
- Algorithms?

Radial-return algorithm



3D plasticity: numerical implementation

- ▶ Numerical resolution: **determination of the plastic strains**

$\underline{\underline{\varepsilon}}^p$: **internal variable**

- ▶ At each time increment, the mechanical state at t_{n+1} :

$$S_{n+1} = \{\underline{u}_{n+1}, \underline{\underline{\varepsilon}}_{n+1}, \underline{\underline{\varepsilon}}^p_{n+1}, \underline{\underline{\sigma}}_{n+1}\}$$

- ▶ has to be calculated by knowing:

$$S_n = \{\underline{u}_n, \underline{\underline{\varepsilon}}_n, \underline{\underline{\varepsilon}}^p_n, \underline{\underline{\sigma}}_n\} \text{ à } t_n$$

- ▶ and the loading:

$$(\underline{f}_{n+1}, \underline{u}_{n+1}^D, \underline{T}_{n+1}^D) \text{ à } t_{n+1}$$

- ▶ **Two particular aspects:**

- ▶ **local resolution scheme**
- ▶ Newton algorithm: **tangent operator** (in order to build the tangent matrix)

3D plasticity: numerical implementation

- ▶ **Local resolution: determination of the plastic strains at the time increment ($n+1$)**

- ▶ generally: backward Euler algorithm

$$\underline{\underline{\varepsilon}}_{n+1}^p = \underline{\underline{\varepsilon}}_n^p + \Delta t F(\underline{\underline{\varepsilon}}_{n+1}^p)$$

- ▶ non linear problem: **Newton(-Raphson)** algorithm

- ▶ **Global resolution: determination of the tangent operator for the creation of the tangent matrix**

- ▶ Stress at time increment ($n+1$) :

$$\underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}_n + \underline{\underline{C}} (\Delta \underline{\underline{\varepsilon}} - \Delta \underline{\underline{\varepsilon}}^p)$$

- ▶ **Tangent operator :**

$$\frac{\partial \underline{\underline{\sigma}}_{n+1}}{\partial \Delta \underline{\underline{\varepsilon}}} = \underline{\underline{C}} : \left(\underline{\underline{I}} - \boxed{\frac{\partial \Delta \underline{\underline{\varepsilon}}^p}{\partial \Delta \underline{\underline{\varepsilon}}}} \right)$$

local “stiffness”

3D plasticity: radial-return algorithm

- Yield function and normality rule:

$$f(\underline{\sigma}) \leq 0 \quad \dot{\underline{\varepsilon}}^p = \lambda \frac{\partial f}{\partial \underline{\sigma}}$$

- Trial elastic stress:

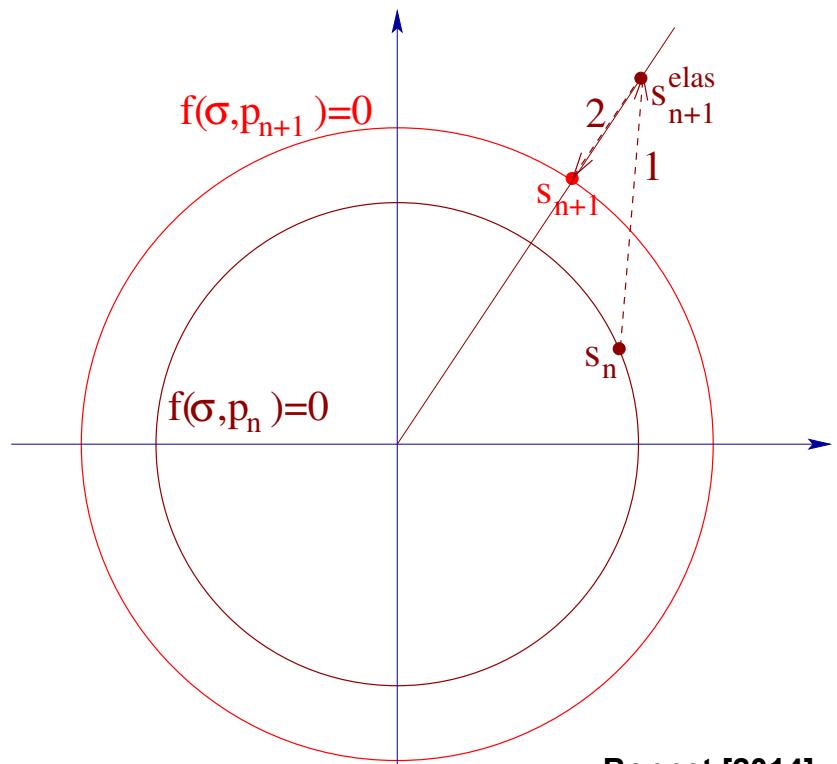
$$\underline{\underline{\sigma}}_{n+1}^{elas} = \underline{\underline{\sigma}}_n + \mathbb{C} : \Delta \underline{\underline{\varepsilon}}$$

- For the solution, let the normal to the convex be:

$$\underline{\underline{N}}_{n+1} = \frac{\underline{\underline{s}}_{n+1}}{\|\underline{\underline{s}}_{n+1}\|}$$

$$\underline{\underline{s}}_{n+1}^{elas} = \left(\|\underline{\underline{s}}_{n+1}\| + 2\mu \Delta p \sqrt{\frac{3}{2}} \right) \underline{\underline{N}}_{n+1}$$

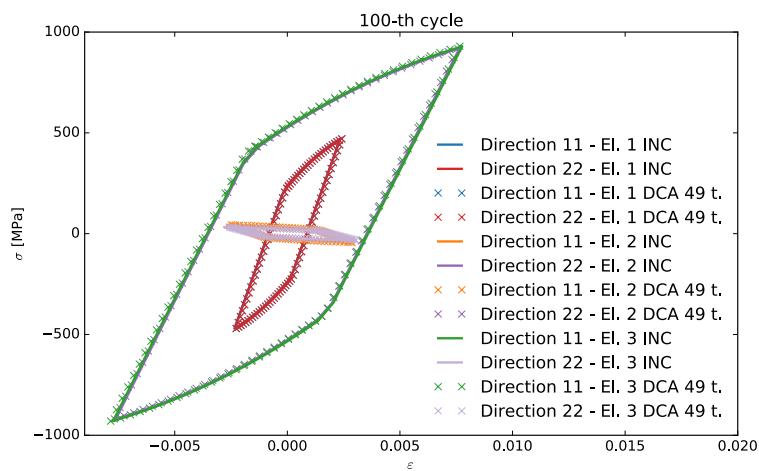
- Solution $\underline{\underline{\sigma}}_{n+1}$: radial projection of the trial stress on the convex at n+1



Bonnet [2014]

Moreau [1971]
Nguyen [1977]
Simo et al. [1985]

Direct algorithms



Some numerical considerations

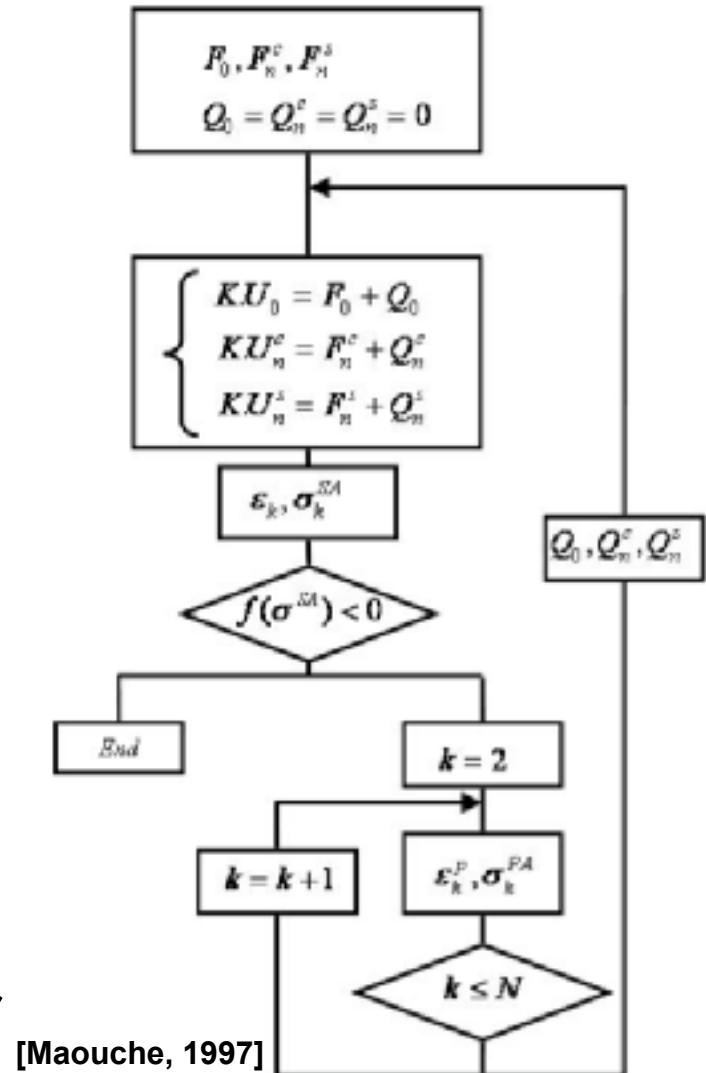
- Asymptotic responses:
 - **stabilization for elastic / plastic shakedown**
 - fatigue criterion: analysis of these stabilized response
- **Can we calculate directly** these asymptotic responses? Yes!
 - direct methods:
 - Direct Cyclic Algorithm
 - Steady-State Algorithm
 - Zarka's method
 - based on the periodicity (stress, strain) of the response: loading Fourier decomposition, shakedown-based analysis, ...



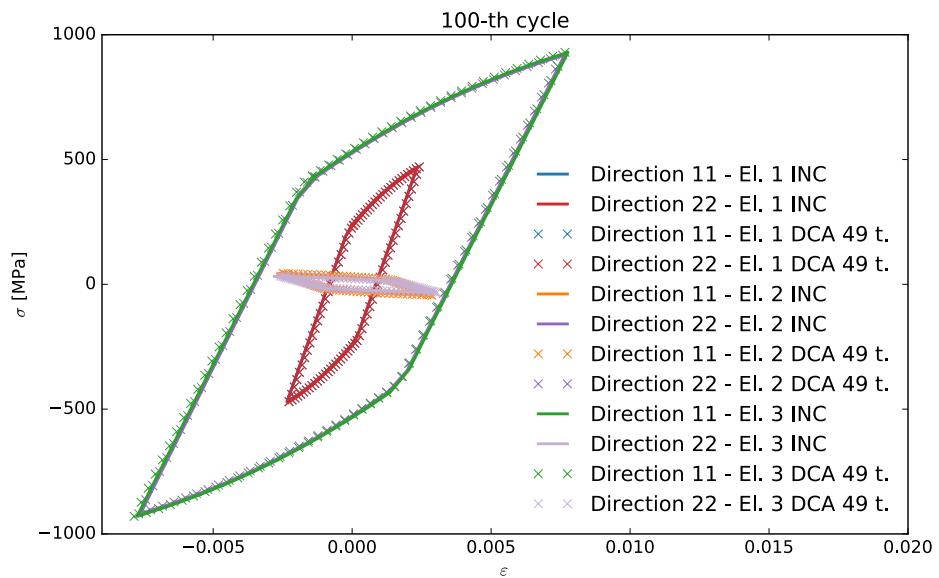
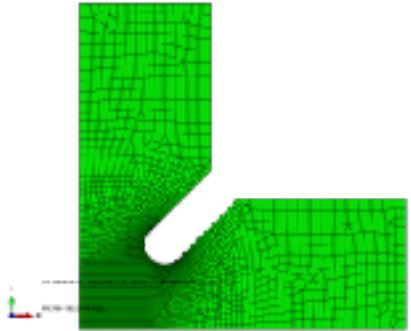
Direct Cyclic Algorithm

Principle:

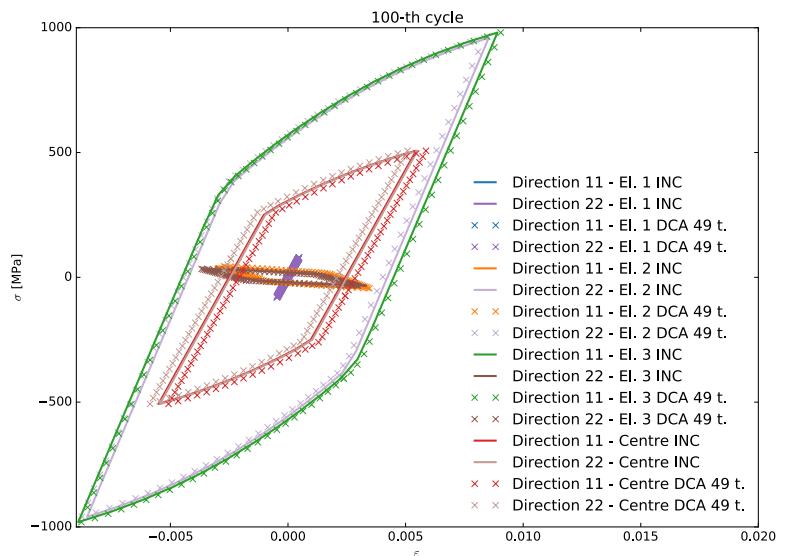
- **Fourier** representation of the displacements, loadings, ... with n terms
- **Global iteration:** Evaluation of the residual vector in m time points
- Fourier representation of the residual vector with n coefficients
- **Modified Newton iterations** to determine corrections to the displacements Fourier terms
- **Local iteration** (see radial-return)
- Enforcement of the **periodicity condition** on the solution (periodic plastic strains and stresses, residual vector tend to zero)



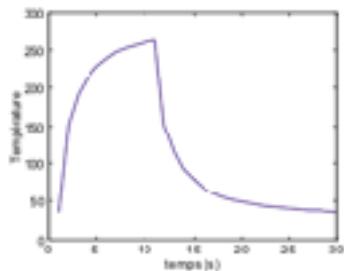
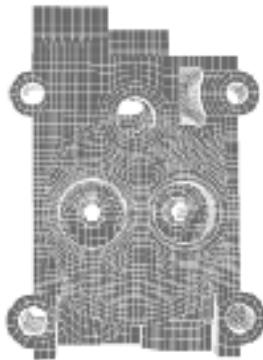
Some comparisons



In-phase biaxial cyclic loading

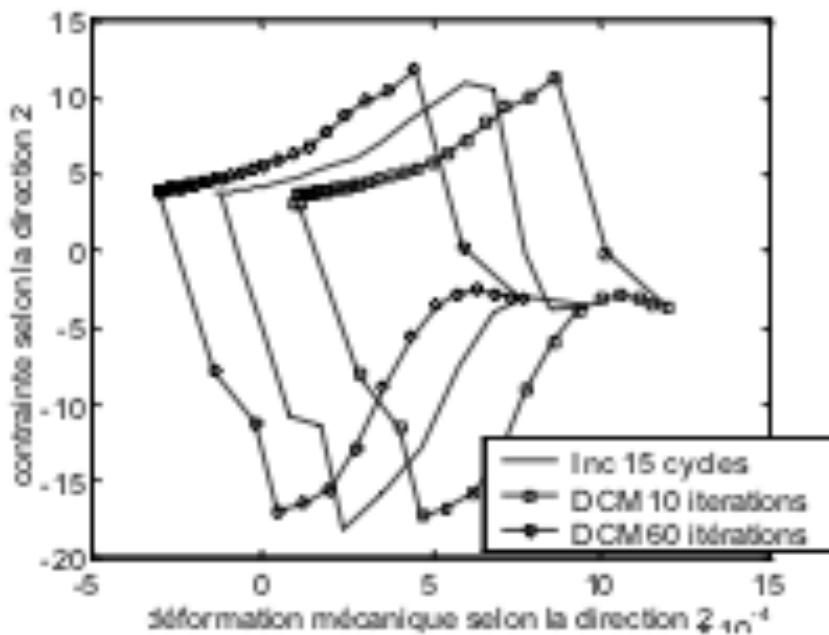
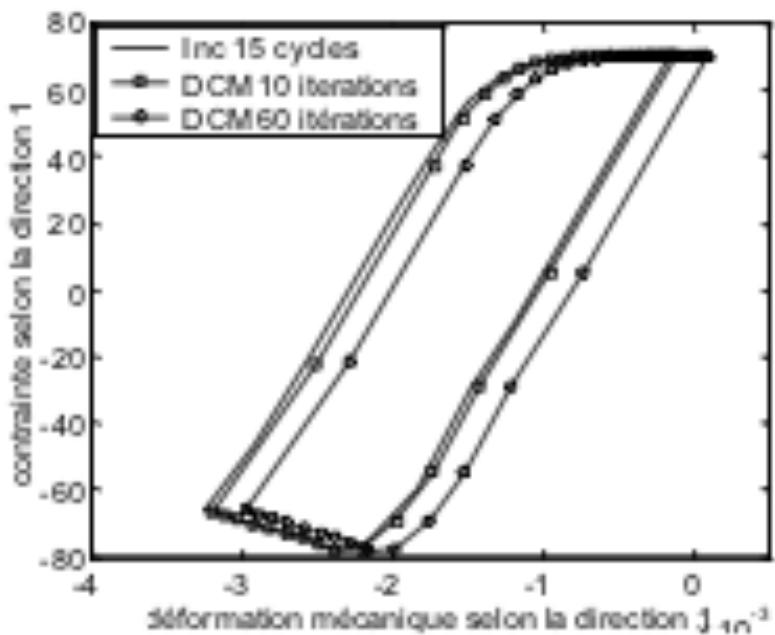


Out-of-phase biaxial cyclic loading



Some comparisons

PSA
GROUPE



[Pommier, 2003]

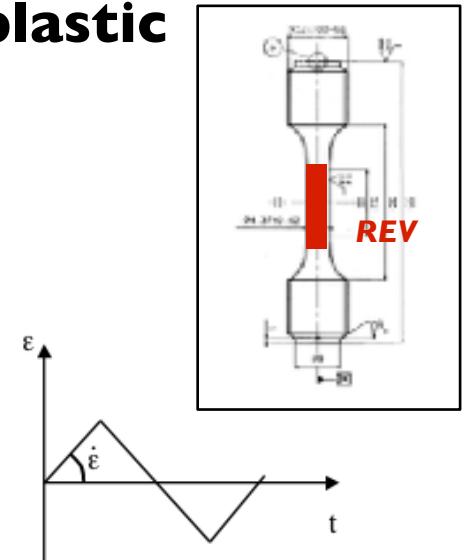
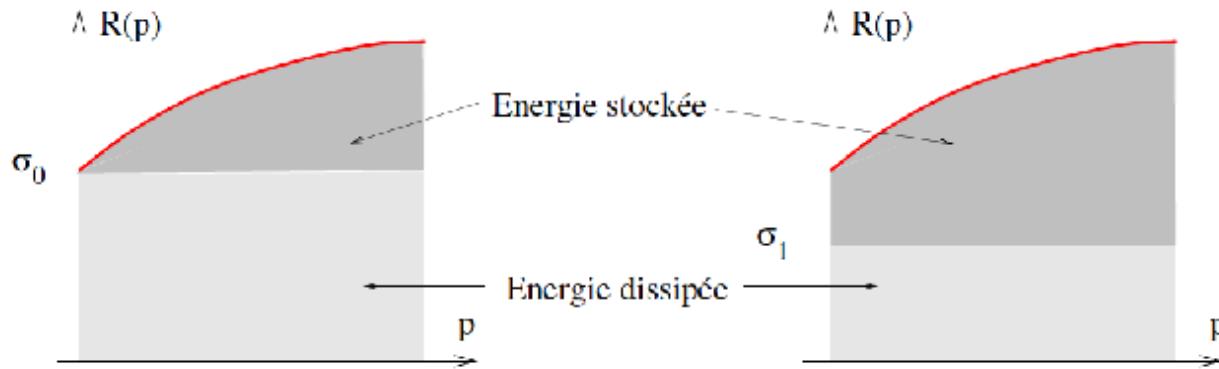
Two scales

- Material scale
 - **yield surface**
 - **strain hardening**
 - Structure scale
 - **structural hardening**
 - **residual stresses**
 - Numerical aspects
 - Material scale : **radial-return**
 - Structure scale : **direct algorithms**
 - Some illustrations, synthesis, remarks and references
- Constitutive models?
- Parameters calibration?
- Algorithms?

Energy balance

- Self-heating tests with **different thermoelastoplastic models**

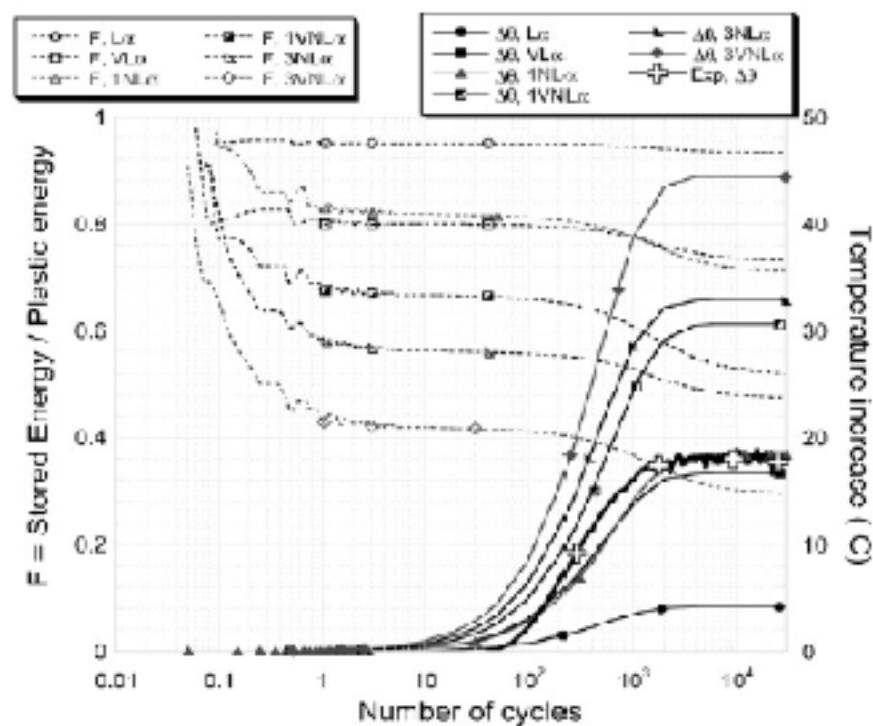
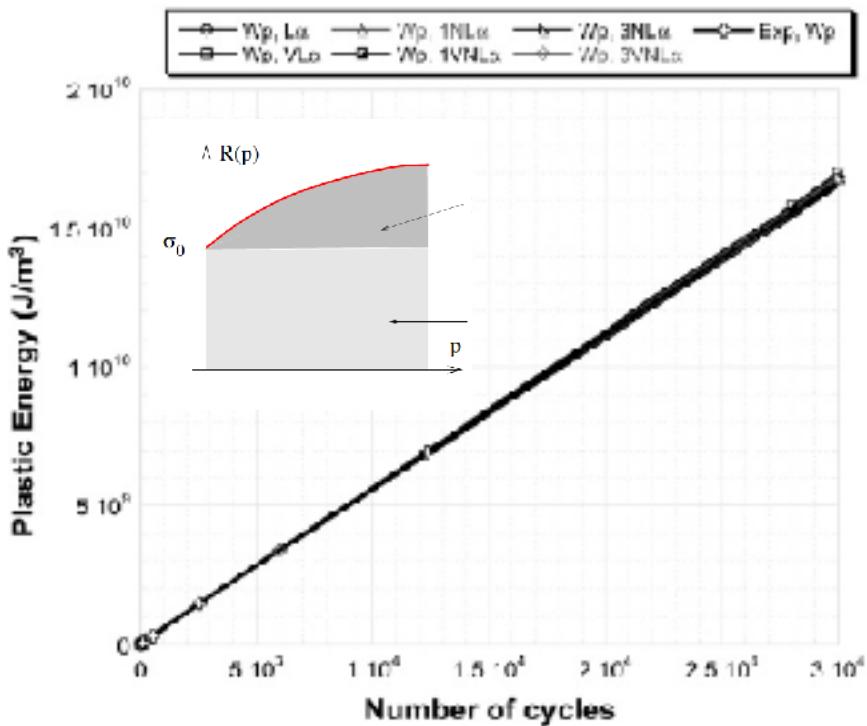
$$\rho C_e \dot{T} + \text{div}(\underline{\underline{q}}) = \underbrace{\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}^p - \underline{\underline{A}_\alpha} : \dot{\underline{\underline{\alpha}}}^e}_{D_1} - T \frac{E\alpha}{1-2\nu} \text{tr}(\dot{\underline{\underline{\varepsilon}}}^e)$$



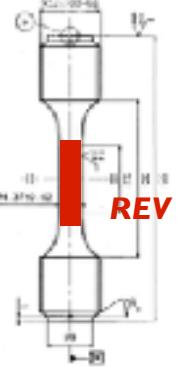
[Vincent, 2008]

Energy balance

- Self-heating tests with **different thermoelastoplastic models**



[Vincent, 2008]



Isotropic vs. Kinematic hardening

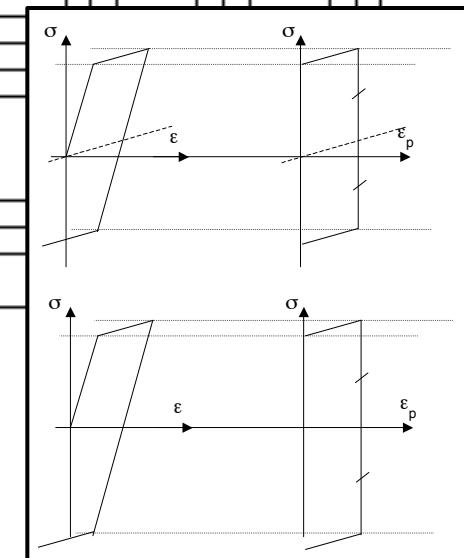
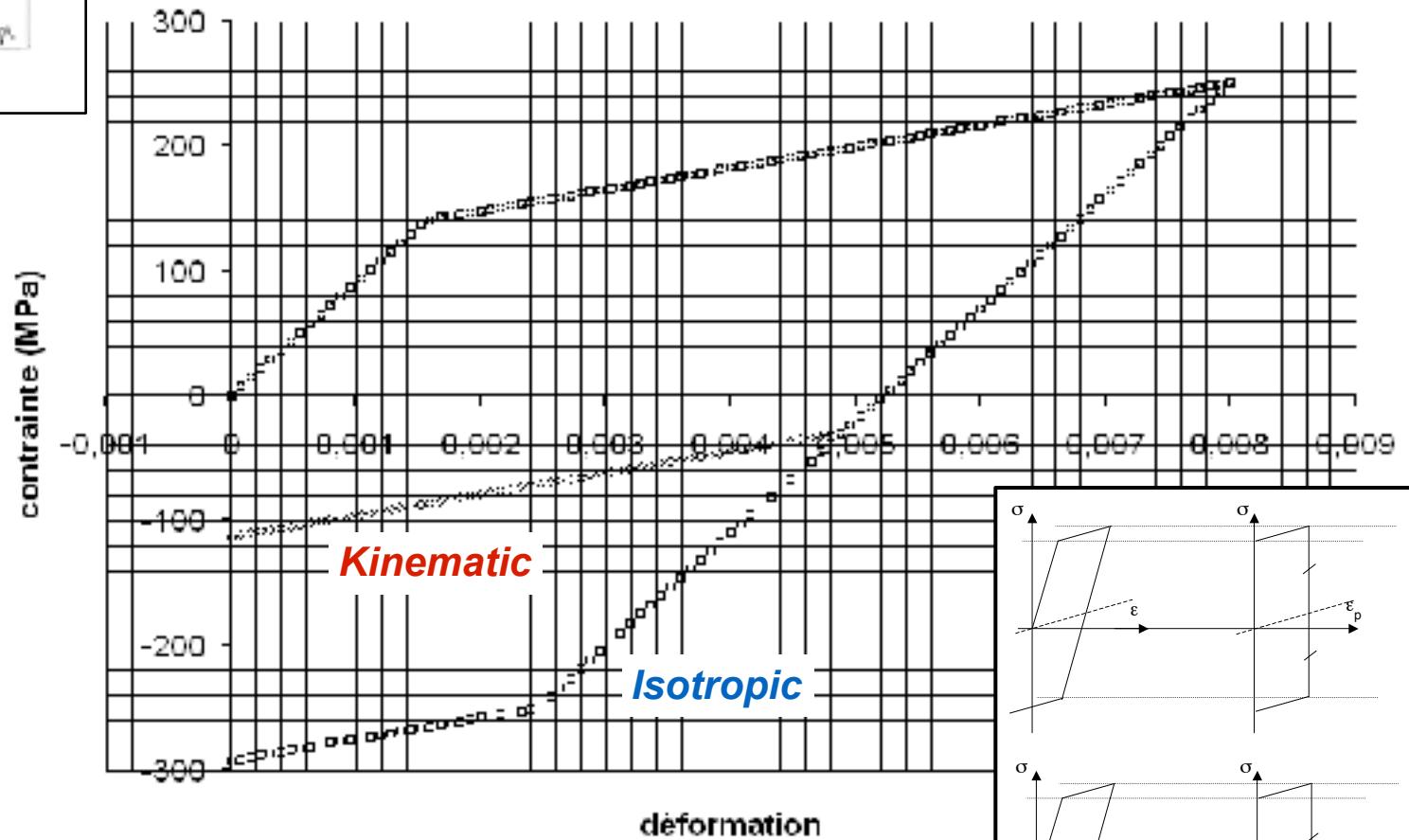
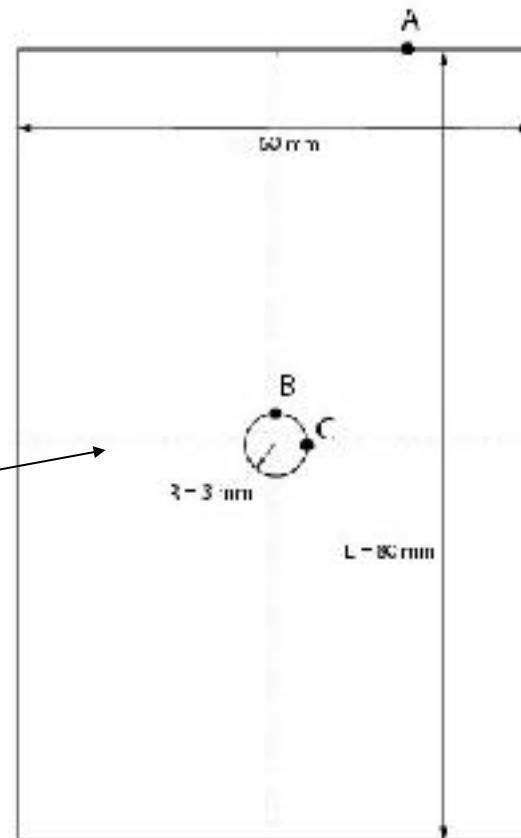
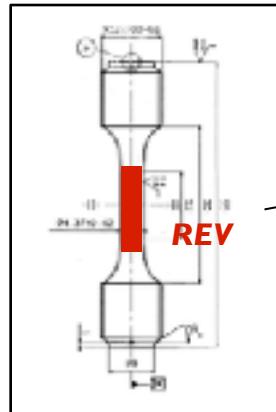
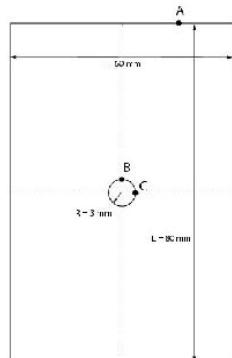


Plate with circular hole

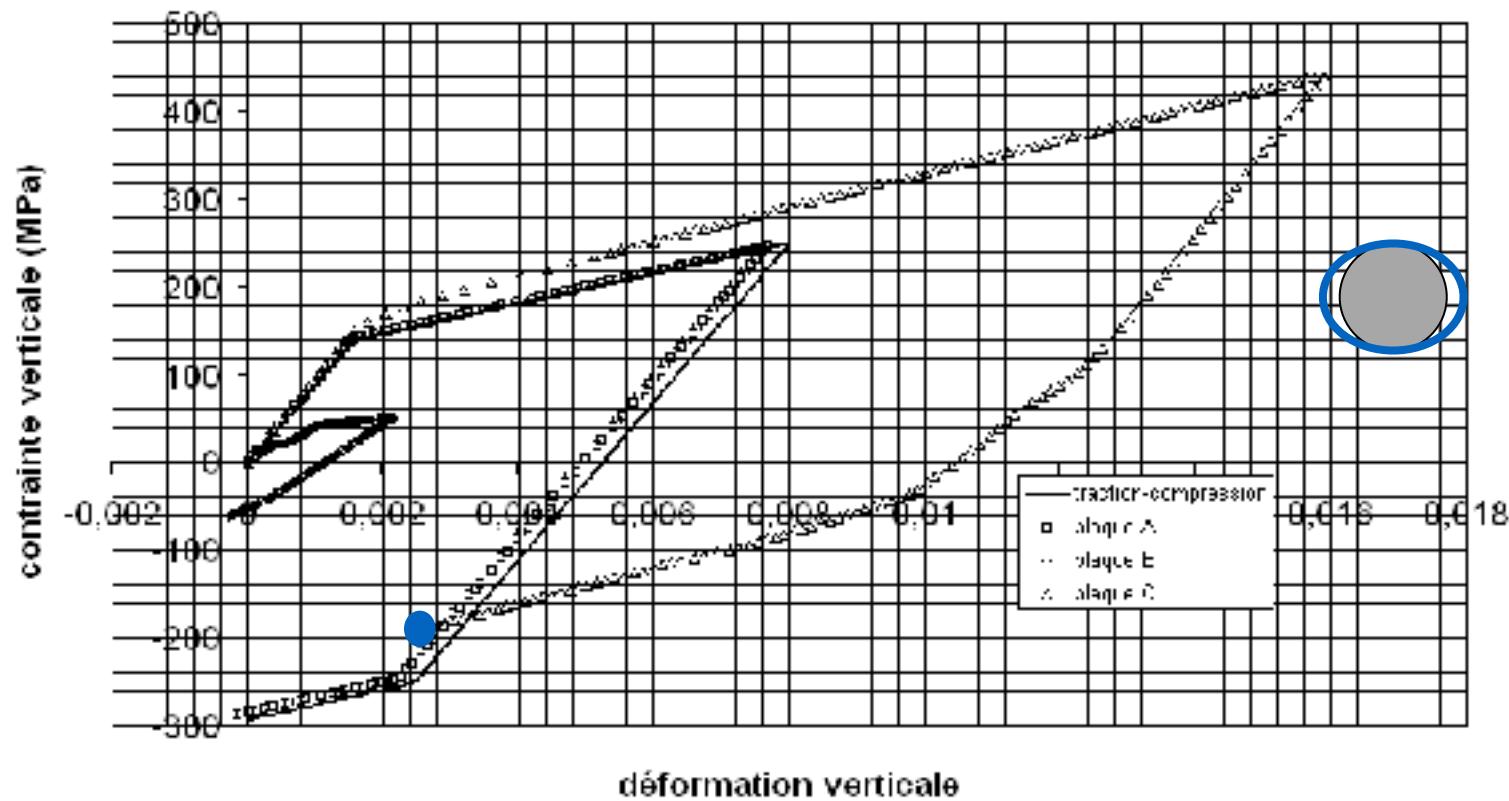
- In the line containing A, displacement-controlled loading-unloading
- Stress-Strain evolutions in B and C:
 - isotropic hardening
 - kinematic hardening

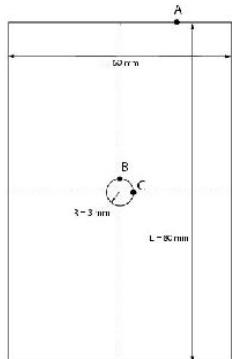




Isotropic hardening

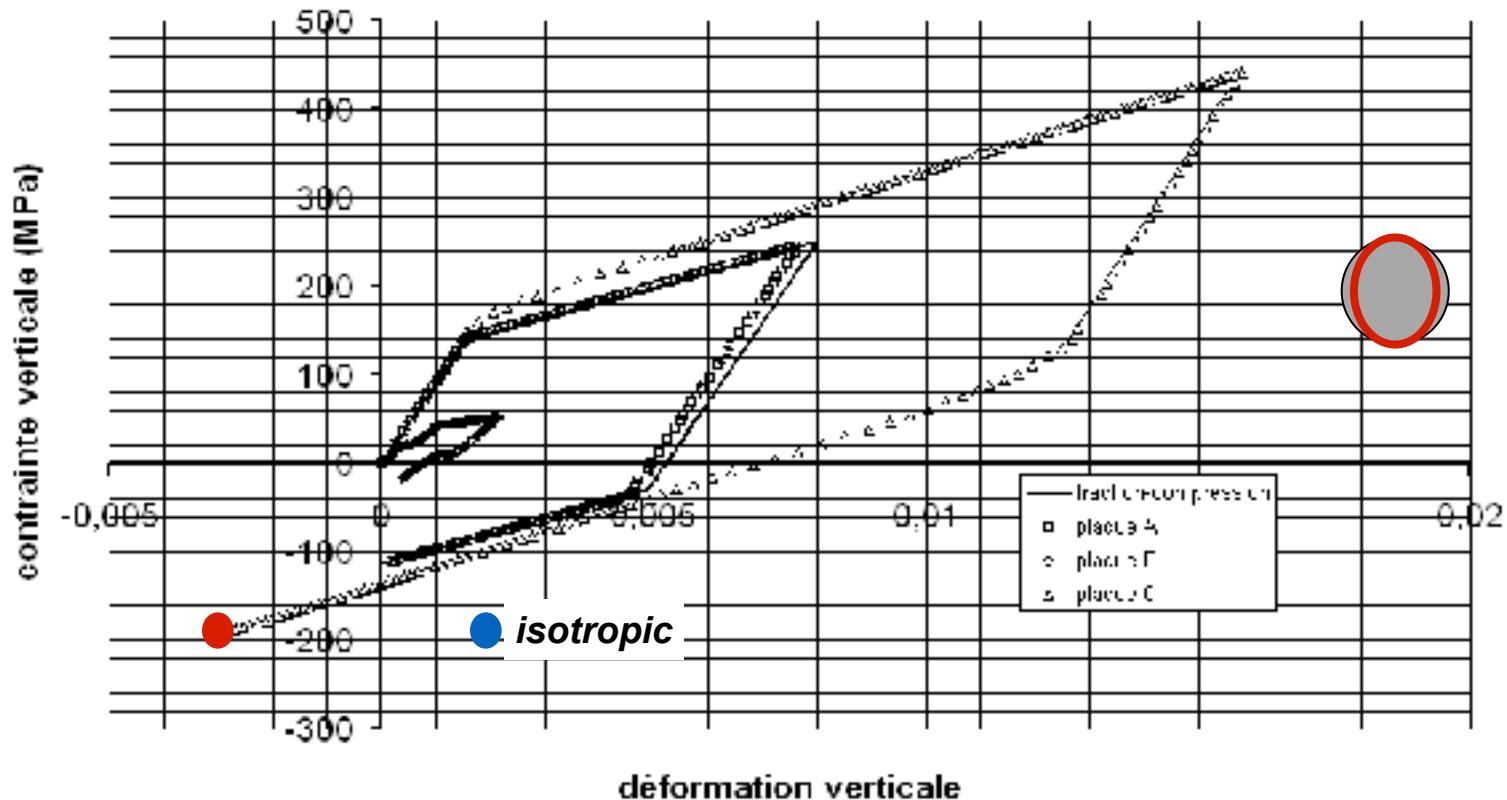
plaqué troué : isotrope



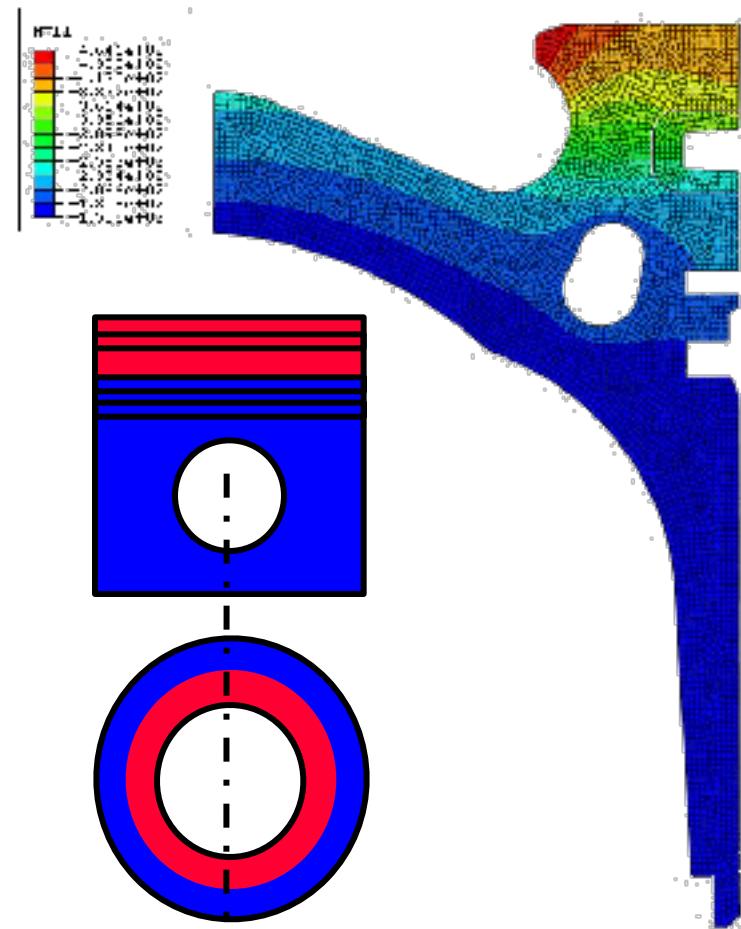
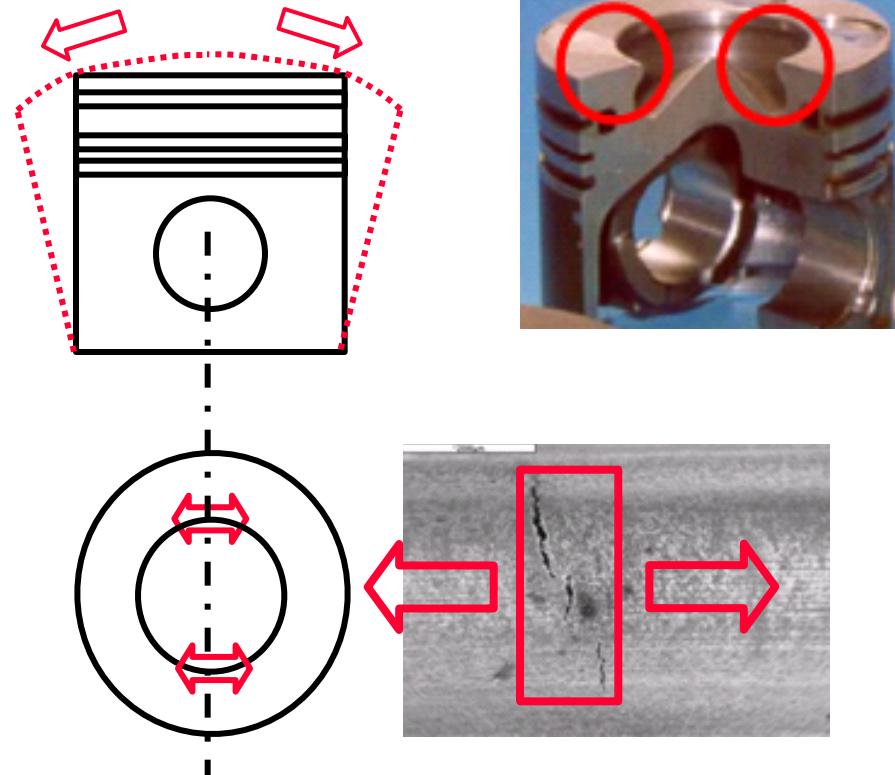


Kinematic hardening

plaqué trouée : cinématique

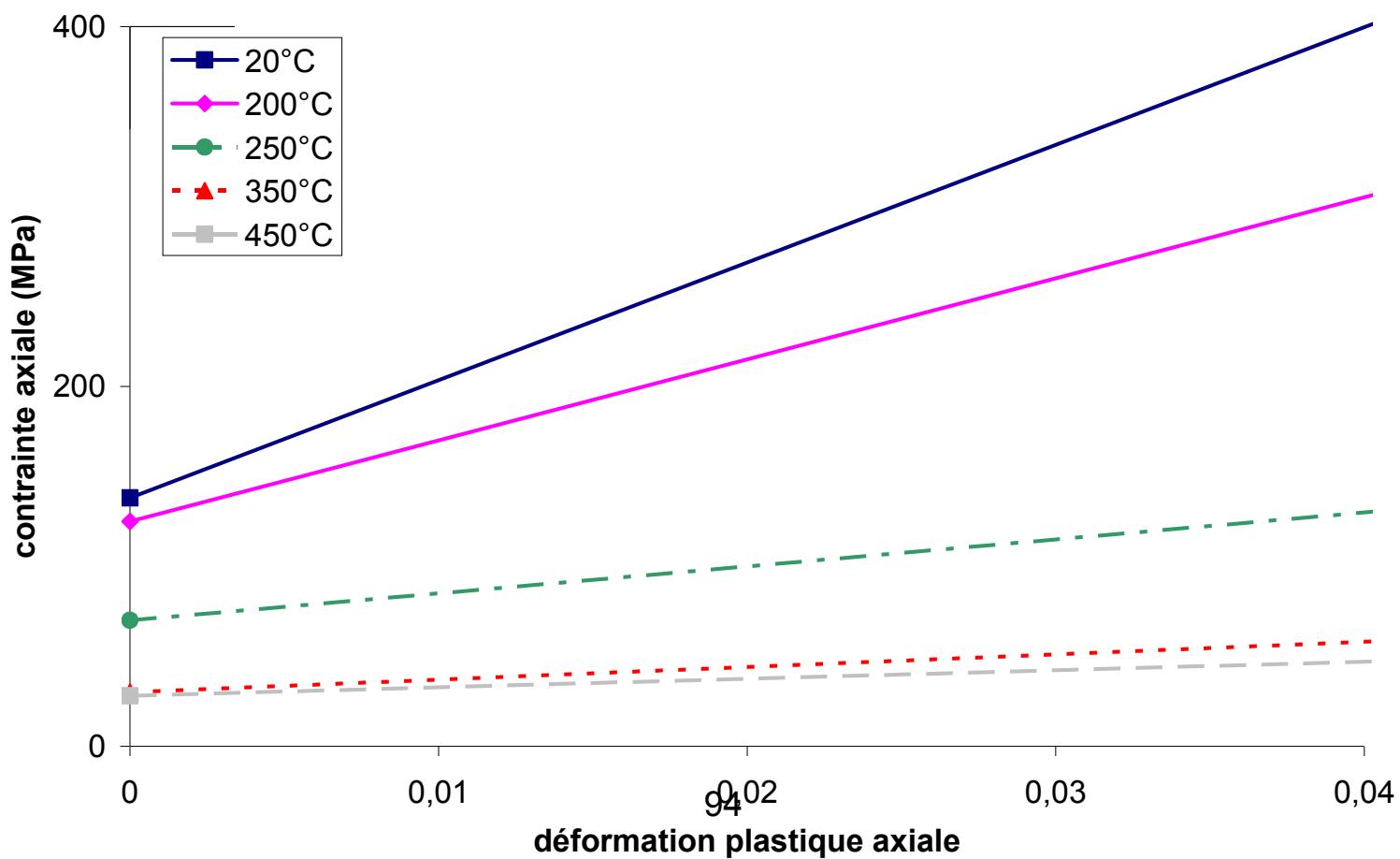


Application to an automotive piston

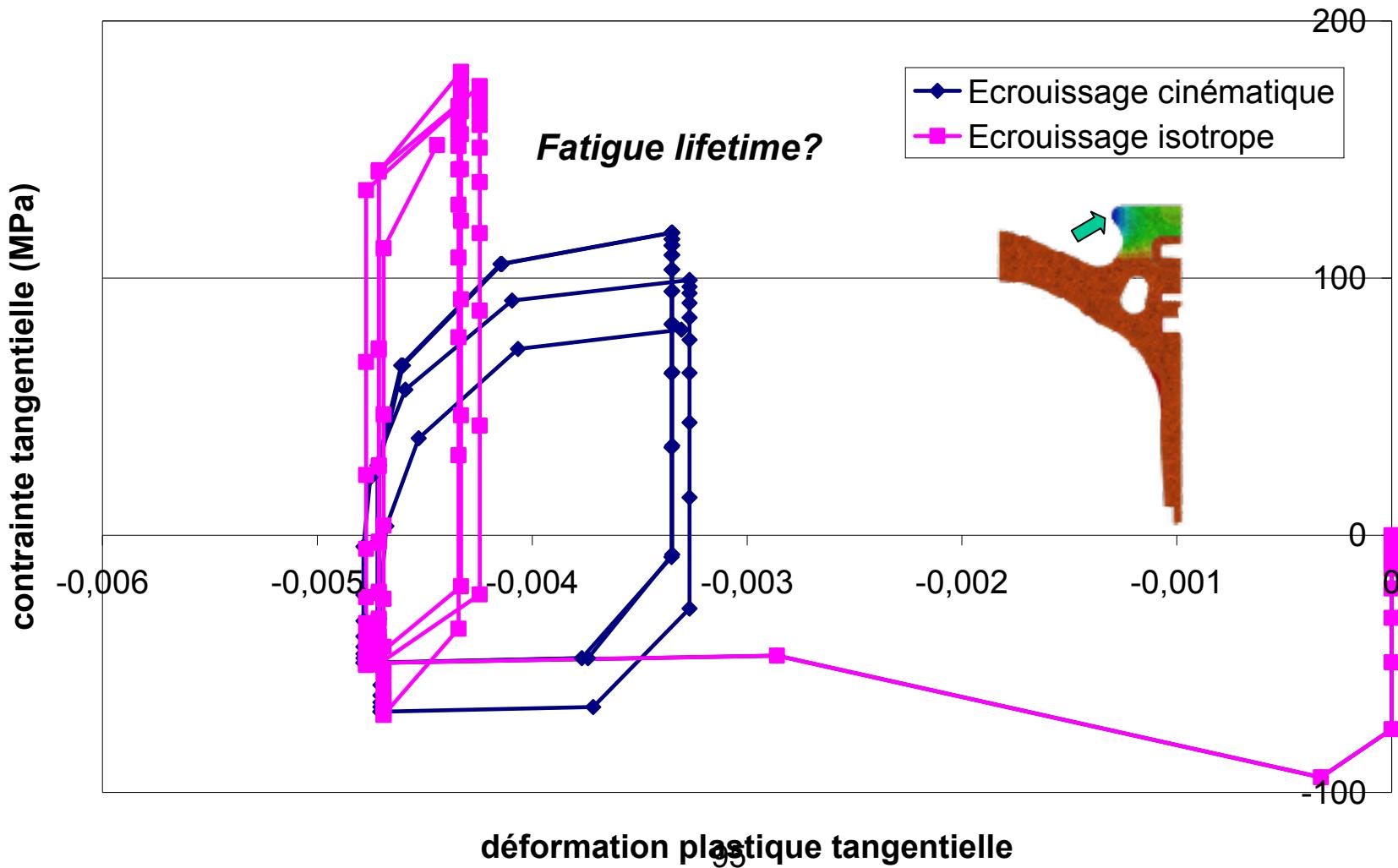


Idealistic mechanical characterization

- Linear hardening plasticity



Influence of the hardening law



Structural effect on energy balance

- Thermoplasticity on structure with **isotropic or kinematic** hardening

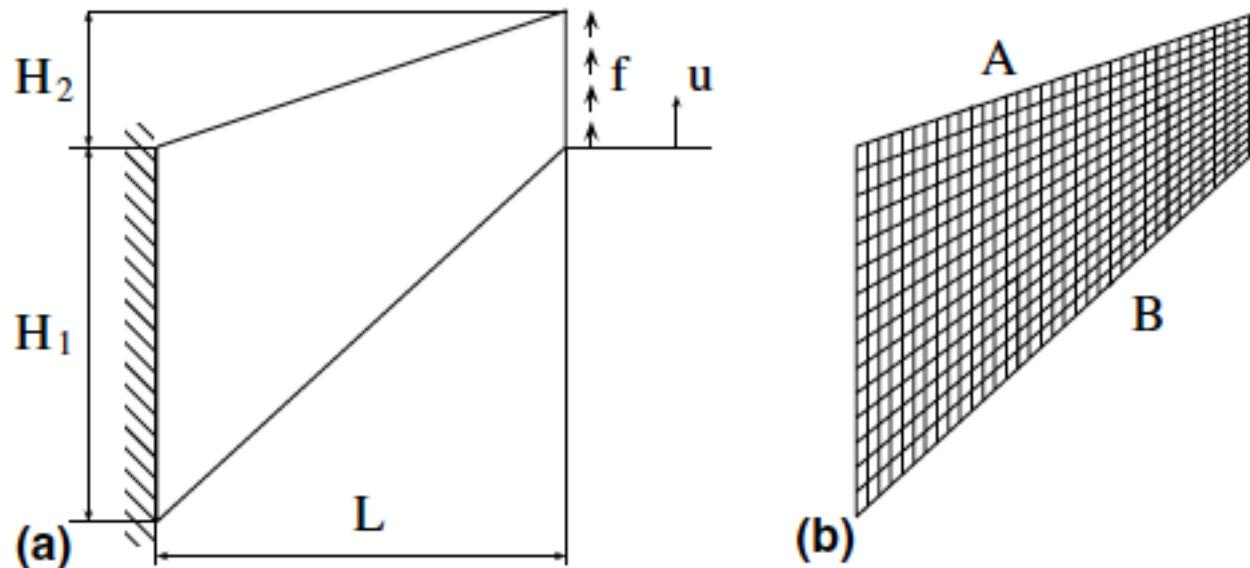


Fig. 9. Cook's membrane: (a) the geometry and (b) the mesh.

[Häkansson et al., 2005]

Energy balance

■ Isotropic hardening

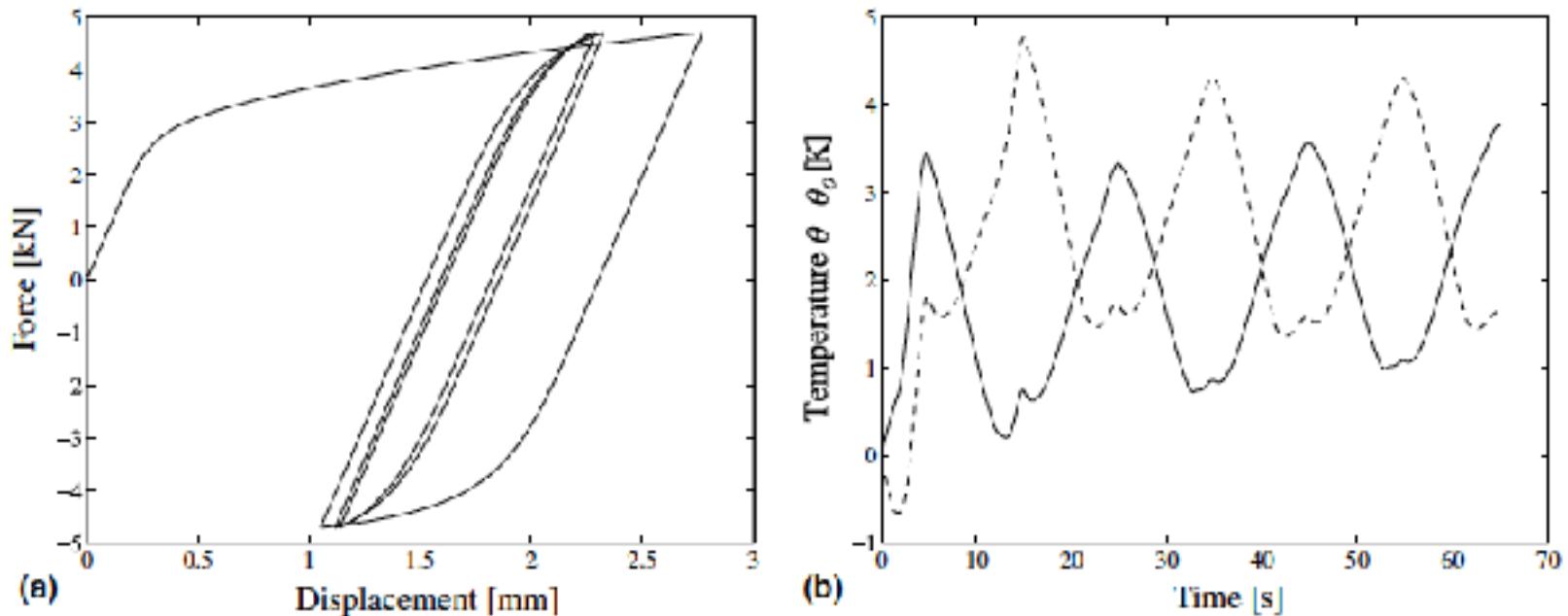


Fig. 11. Cyclic thermoelasto-plastic response for the isotropic hardening model. (a) Mechanical response; (b) temperature evolution versus time, solid line corresponds to point A and the dashed one to point B, cf. Fig. 9(b).

[Håkansson et al., 2005]

Energy balance

Kinematic hardening

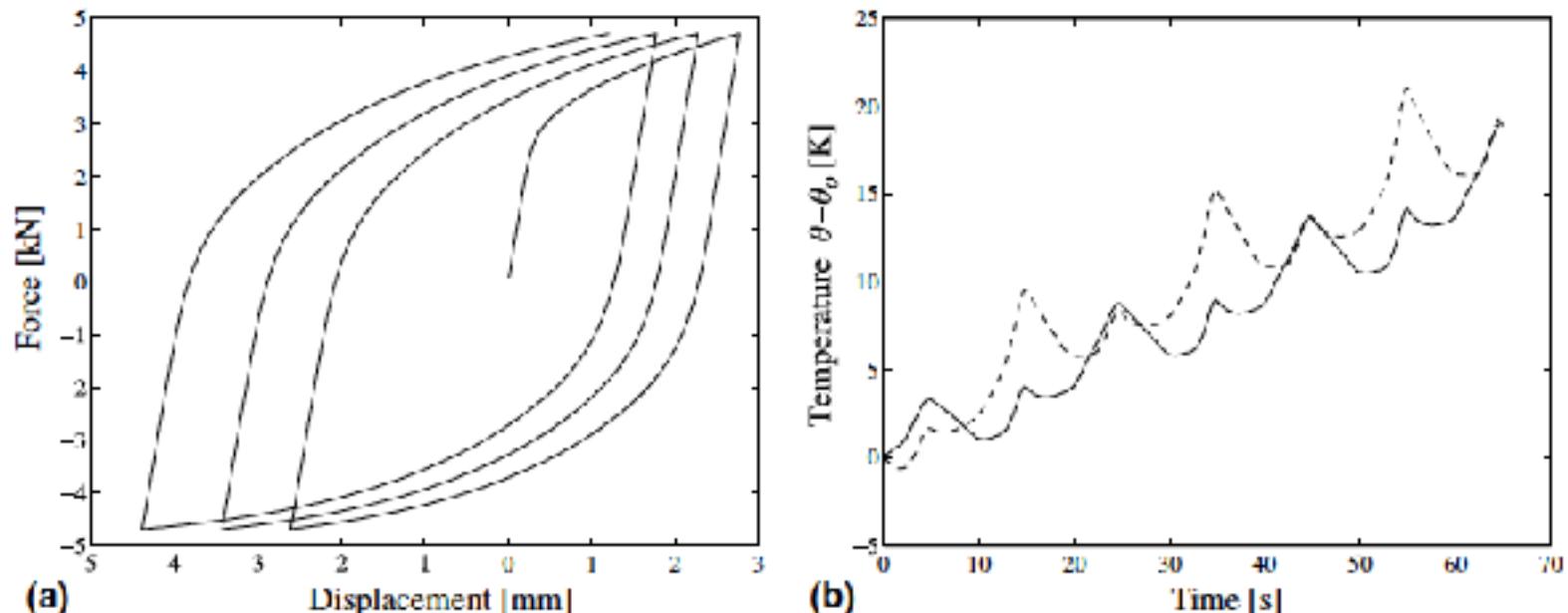


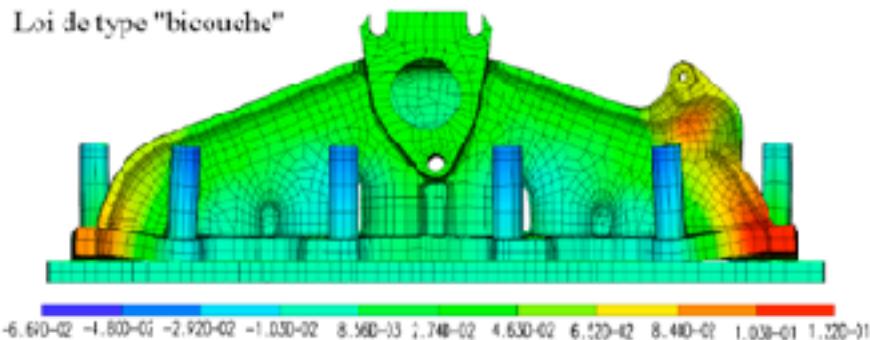
Fig. 10. Cyclic thermoclasto-plastic response for the kinematic hardening model. (a) Mechanical response; (b) temperature evolution versus time, solid line corresponds to point A and the dashed one to point B, cf. Fig. 9(b).

[Håkansson et al., 2005]

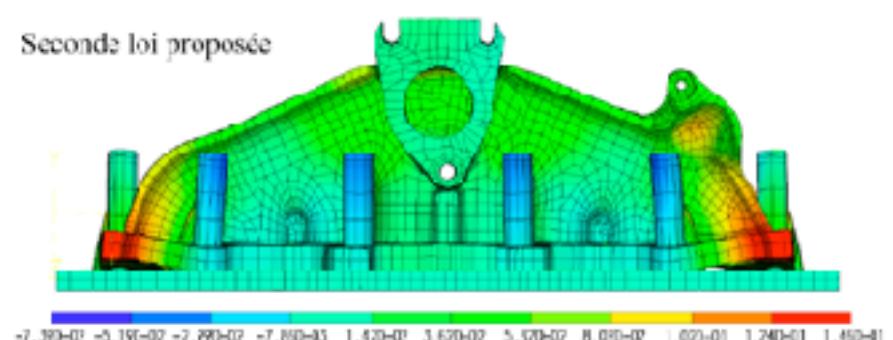
Residual strain and deformation

- Exhaust manifolds under thermal shock: **residual deformation** with 4 different viscoplastic laws

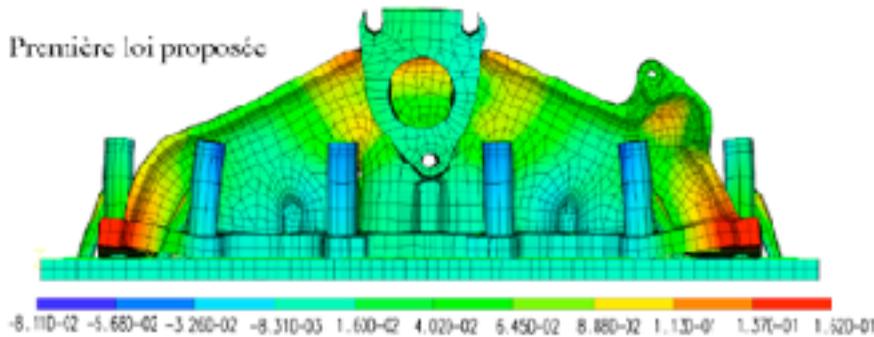
Loi de type "bicouche"



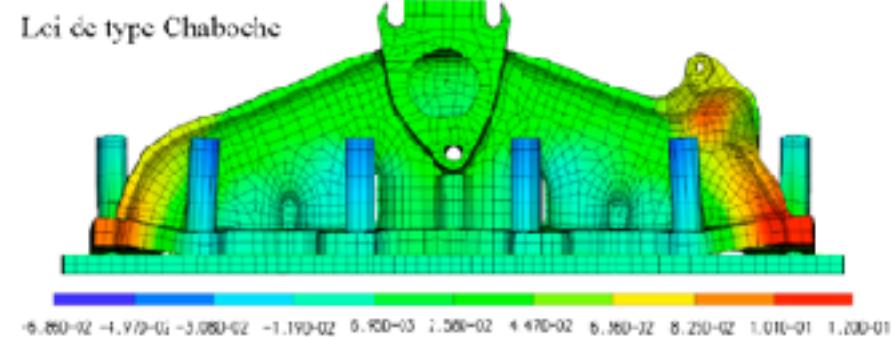
Seconde loi proposée



Première loi proposée



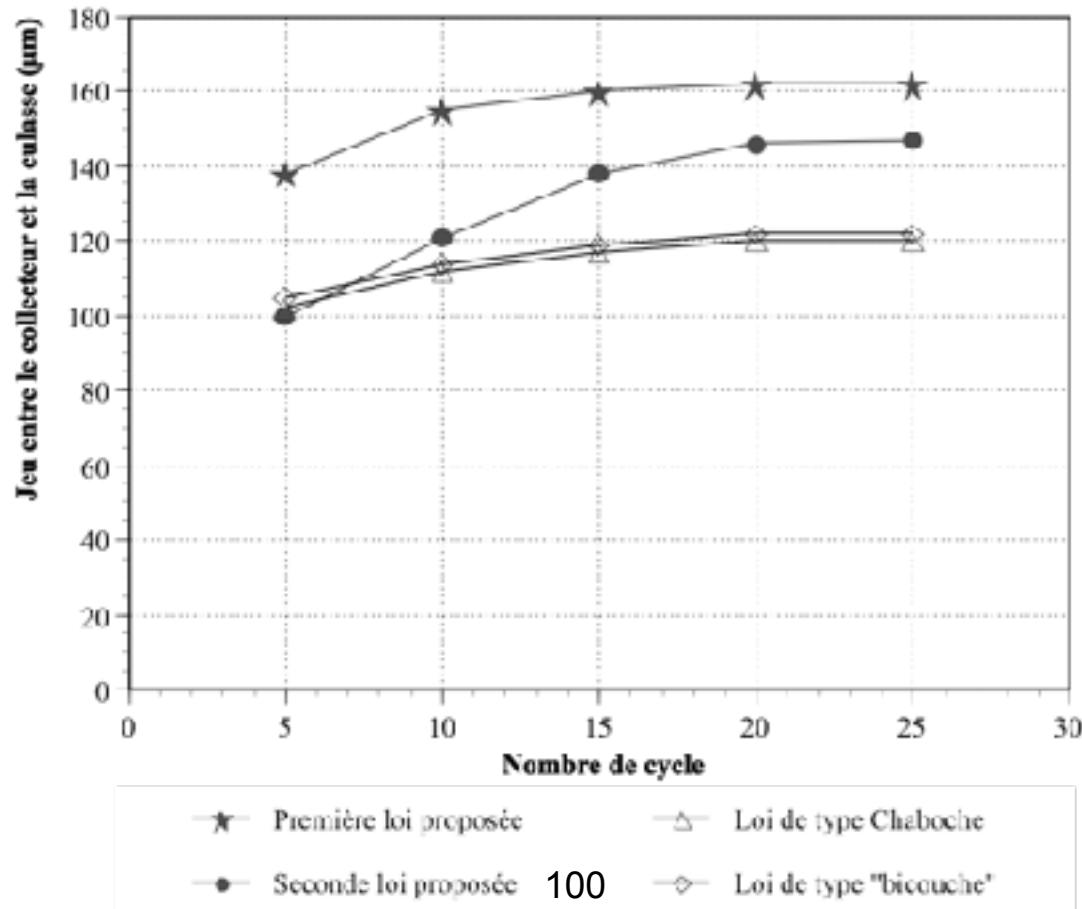
Lei de type Chaboche



[Szmytka, 2007]

Residual strain and deformation

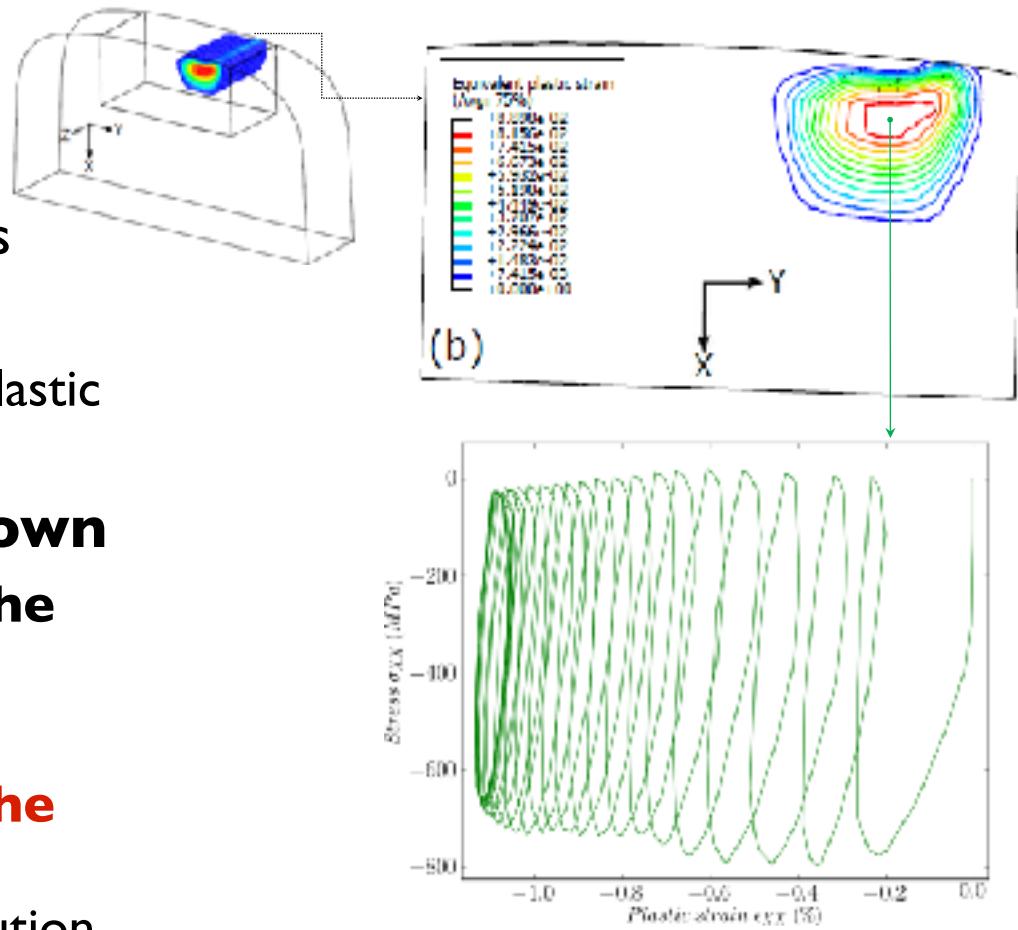
- Exhaust manifolds under thermal shock: **residual deformation** with 4 different viscoelastic laws



[Szmytka, 2007]

Structure and material scales: coupling

- Contact pressure:
 - maximum shear stress under the surface
- Elastic-plastic simulations
 - first rolling cycles
 - asymptotic response: plastic shakedown?
- Delicate point: **shakedown**
 - highly **depends on the plastic hardening** constitutive law
 - highly **depends on the loading evolution**
(contact pressure evolution due to plasticity!)



[Saint-Aimé, 2017]

Synthesis

- ▶ **Material scale:**

- ▶ **Yield function** (example: von Mises): **when does plasticity occur?**
- ▶ **Flow rule** (example : normality rule): **how does plastic flow occur?**
 - without volume changes: **deviator** (counterexample : soil mechanics)
 - with or without **hardening**: evolution of the yield surface
 - **Isotropic**: size increases
 - **Kinematic** : yield surface translation in the deviator space
- ▶ Permanent strain: **plastic strain**
- ▶ Numerical implementation: **radial-return algorithm**

- ▶ **Structure scale:**

- ▶ Incompatibilities: **structural hardening** and **residual stresses**
- ▶ **Direct computations** are possible (with more or less strong assumptions)

Remarks

- Elastoplastic constitutive laws:
 - **Combined hardening rules:** « classical » but influence on structure scale (ratcheting, energy balance, residual stress and deformation)
 - Parameters calibration: non uniqueness! Need a large cyclic database and/or ... thermomechanical data for **energy balance!**
 - Many constitutive laws in **FE codes**, in other way UMat, Z-front, MFront, ... (see references)
- Extension to **viscoplasticity**:
 - Viscoplastic potential regularizes the problem
 - Same type of hardening laws including **recovering**

Viscoplasticity

■ Example:

- Yield function:

$$f = \sqrt{3J_2} - \sigma_y$$

- Flow rule:

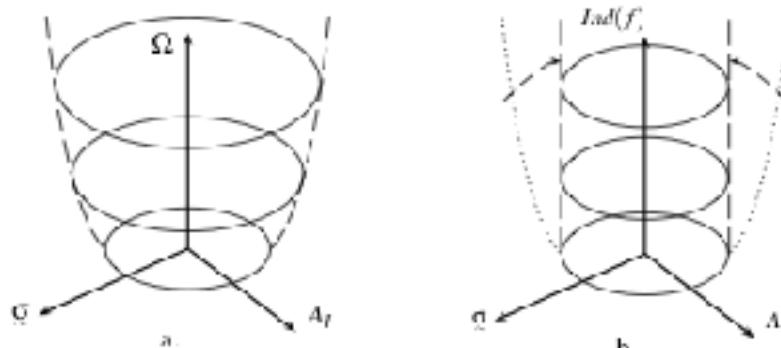


FIG. 4.1 – Comparaison des théories de plasticité et de viscoplasticité. (a) potentiel viscoplastique. (b) obtention d'un modèle plastique par passage à la limite

[Cailletaud et al., 2009]

$$\dot{\varepsilon}_{vp} = \sqrt{\frac{3}{2}} \left\langle \frac{J_2(\boldsymbol{\sigma} - \boldsymbol{X}) - \sigma_y}{\eta} \right\rangle^m \frac{\text{dev}(\boldsymbol{\sigma} - \boldsymbol{X})}{J_2(\boldsymbol{\sigma} - \boldsymbol{X})}$$

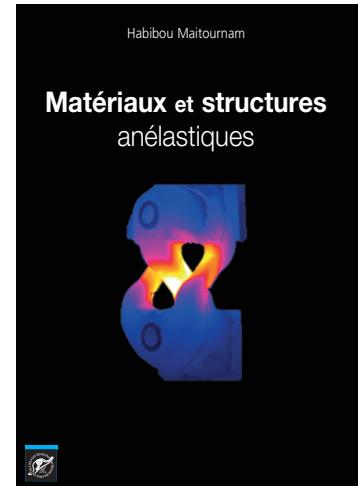
- Certainly more elements ...

- Fabien Szmytka on cylinder heads,
- N. Saintier, L. Signor and C. Mareau on polycrystals in fatigue
- V. Maurel on crack growth in generalized plasticity
- S. Pommier on incremental crack growth
- P. Kanoute on stress gradient
- S. Fouvry on fretting fatigue
- ...

(My) References

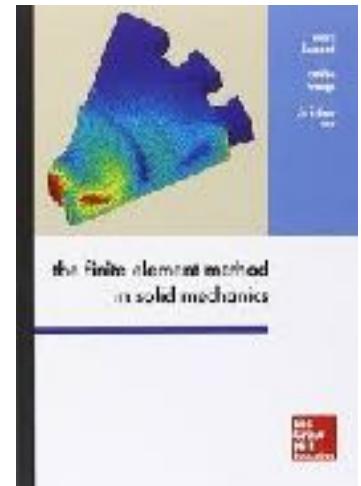
■ Ecole Polytechnique:

- Plasticité (MEC 551)
 - Pierre Suquet then Marc Bonnet (until 2010):
 - <http://perso.ensta-paristech.fr/~mbonnet/mec551/mec551.pdf>
 - Jean-Jacques Marigo (from 2010):
 - <https://cel.archives-ouvertes.fr/cel-00549750v2/document>
- Matériaux et Structures anélastiques (MEC 562)
 - Habibou Maïtournam
- Analyse des structures mécaniques par la méthode des éléments finis (MEC 568)
 - Marc Bonnet



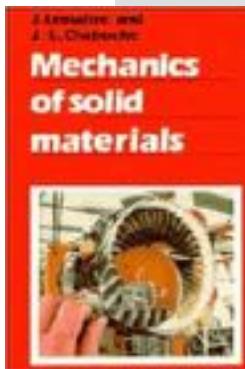
■ Ecole des Mines de Paris:

- Mécanique des Matériaux Solides (3122)
 - Georges Cailletaud, ...
 - http://mms2.ensmp.fr/mms_paris/mms_Paris.php

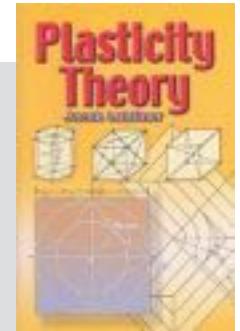


References

► Classical and useful:



- ▶ **J. Lemaitre, J-L. Chaboche**, Mechanics of Solids Materials, Cambridge University Press, 1994
- ▶ **J. Lubliner**, Plasticity theory, Dover editions, 2008
[http://www.ewp.rpi.edu/hartford/~ernesto/F2008/MEF2/Z-Links/Papers/
Lubliner.pdf](http://www.ewp.rpi.edu/hartford/~ernesto/F2008/MEF2/Z-Links/Papers/Lubliner.pdf)
- ▶ **J. Simo and TJR Hughes**, Computational inelasticity, Springer, 1998

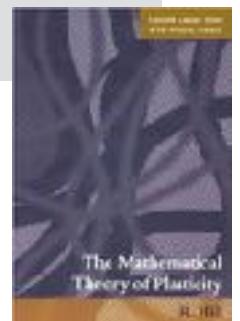


► Historical :

- ▶ **R. Hill** :The Mathematical Theory of Plasticity (1950), new edition, Oxford, 1998.



R. Hill
(1921-2011)



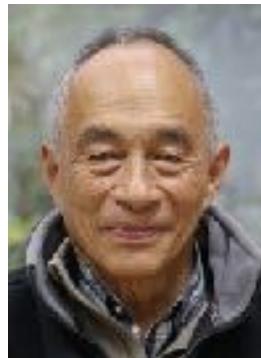
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Some questions?

